THEORY OF OSCILLATIONS

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OF
OSCILLATIONS

By A. A. ANDRONOW
and C. E. CHAIKIN

English Language Edition
Edited under the direction of
SOLOMON LEFSCHETZ

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PREFACE TO THE ENGLISH LANGUAGE EDITION

The present work is a condensed version of one of the same title, published in 1937 in Russian and containing the first extensive treatment of non-linear oscillations. Messrs. Andronow and Chaikin are both members of the Institute of Oscillations founded about a decade and a half ago by the late Soviet physicist L. I. Mandelstam. The work of this Institute has become known to the world at large through Dr. N. Minorsky's *Introduction to Non-Linear Mechanics* recently issued by the David Taylor Model Basin. The great interest evoked by this excellent report provided a strong incentive for making available to the scientific and technical public the text of Andronow and Chaikin. This has been made possible with the aid of the Office of Naval Research as part of the project on Non-Linear Differential Equations which it is sponsoring at Princeton University.

The theory of harmonic oscillations, or oscillations of sinusoidal type, is well known. It is also quite elementary, since it is based upon *linear* differential equations whose explicit solutions are readily obtained. Unfortunately nature refuses to remain linear and repeatedly presents us with *non-linear* oscillations, that is to say with oscillations based upon non-linear differential equations. It is often possible to *linearize* a problem, i.e. to modify the operating assumptions so as to have a linear situation without causing important deviations. Thus the true equation of the pendulum is never linear, but for very small deviations (a few degrees) it may be satisfactorily replaced by a linear equation. In other words for small deviations the pendulum may be replaced by a harmonic oscillator. This ceases to hold however for large deviations, and in dealing with these one must consider the non-linear equation itself and not merely a linear substitute.

There are then valid practical reasons for an extensive study of non-linear oscillations. Since their equations rarely admit an explicit solution, one will be forced to have recourse to various mathematical doctrines to obtain all possible information. The text of this book aims precisely to provide an introduction to this general subject.

Much attention is paid in the book to self-excited oscillations, a type of self-sustained oscillations into which some physical systems tend to go. These oscillations may be most valuable as in the Froude pendulum, a clock, a vacuum tube circuit, or they may be parasitic and even harmful as in ship stabilizers, Prony brakes, etc. For one reason

or the other it is clear that their study is becoming imperative and in this connection the present volume will be found most useful.

In general plan the book takes up in the first four chapters the theories which require the least mathematical preparation. The fifth chapter is mainly devoted to the presentation of the basic results of Poincaré and Liapounoff, without which no real progress is possible. These results are then applied in various ways in the remaining chapters.

For the convenience of the reader there is a moderate bibliography of the most accessible books and papers. There has also been provided a reference list of the numerous practical applications treated in the text. A perusal of the list will show that they bear mainly upon vacuum tube circuits. This is due to the very practical reason that vacuum tube experiments are far more convenient and economical than any others and thus most suitable as a check on the mathematical theory.

The translation from the original Russian is due to Dr. Natasha Goldowskaja, and the general editorial work was carried out under the direction of the undersigned, aided by Drs. J. P. LaSalle, L. L. Rauch, C. E. Langenhop, and A. B. Farnell. It became evident quite early that considerable condensation, paring down and editing of the original was unavoidable if its value were not to be lost to the non-Russian reader. On repeated occasions, especially in the subject matter of Chapter V, the authors had developed a mathematical machinery going much beyond the possible applications. In all such cases the theoretical material has been scaled down to the "visible" practical requirements. For the same reasons many lengthy and purely theoretical discussions have likewise been reduced or entirely suppressed. The suppressed material is either familiar to our readers or else available in readily accessible publications. On the other hand the numerous practical examples which constitute one of the major attractions of the book have been scarcely touched, and generally only to "smooth out" the mathematical treatment. Thanks to the excisions and modifications the book has been reduced considerably in size and at the same time made much more readable. It is hoped that in spite of this we have preserved all the flavor of a most interesting and important contribution to the literature on oscillatory phenomena.

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THEORY OF OSCILLATIONS

Introduction

In every theoretical treatment of a physical phenomenon, considerable idealization of the true properties is unavoidable. One may classify a system as linear, conservative or otherwise, as a system with so many degrees of freedom, etc. All such assertions are idealizations. Real systems are neither linear nor conservative, nor do they have a finite number of degrees of freedom since they cannot be described with absolute precision by mathematical relations. Therefore, a strict classification can be applied only to abstract schemes obtained from appropriate idealizations.

The selection of the idealization and the elements to be omitted or underscored depend in each case upon the questions to be discussed. Take, for instance, a steel ball falling vertically on a steel plate. If we are mainly interested in the movement of the ball as a whole, we consider it as a material point moving under its own weight and whose velocity is reversed when the ball hits the plate. On the other hand, if we are interested in the elastic stresses caused by the shock in the ball, we must consider the ball as an elastic body with the characteristics of steel, take into account the nature of the deformation, the duration of the collision, etc. Similarly, upon neglecting friction or other dissipative forces, a system will be idealized as conservative. Needless to say, one must often consider dissipation and give up conservativeness.

The usual idealizations of the simpler dynamical or electrical systems lead naturally to systems of differential equations. In systems of one degree of freedom the basic differential equation is of the second order. If possible the idealization is so selected that the equation is linear and with constant coefficients. A typical equation is the equation of motion of a mass under the action of a spring with friction and exterior force:

$$m\ddot{x} + 2h\dot{x} + kx = F(t) \tag{1}$$

where here and throughout this book the dots denote time derivatives (Newton's notation). Similarly the current i in a so-called R,L,C circuit is governed by

$$L\ddot{i} + R\dot{i} + \frac{1}{C}i = \frac{dE(t)}{dt} \tag{2}$$

where $E(t)$ is the impressed e.m.f. If possible one will select the situation so that in (1) the coefficients m,h,k and in (2) the coefficients L,R,C are constant. However, let us suppose that in (1) $F = 0$ or in (2) E is constant, so that the equations are homogeneous. Unless special assumptions are introduced, one will have $h \neq 0$ in (1) or $R \neq 0$ in (2). In natural systems this will always be the case. We will then have a system of the form

$$(3) \qquad a\ddot{x} + b\dot{x} + cx = 0, \qquad abc \neq 0.$$

Now, as is well known, in such systems no permanent oscillations (strict periodic motions) can arise. There are, however, well-known physical systems, for instance circuits with vacuum tubes, in which there do arise permanent oscillations; the slightest disturbance starts one going. These systems cannot therefore be governed by equations of type (3), and they force us to idealizations in which one or more of the coefficients a,b,c depend on x or $\dot{x}$ or both. In other words, the basic differential equation must be non-linear. Thus the theory of the vacuum tube leads to the well-known equation of van der Pol

$$(4) \qquad \ddot{x} + \mu(x^2 - 1)\dot{x} + x = 0, \qquad \mu > 0,$$

one of the simplest non-linear equations to be dealt with in the sequel. In general terms we may state our basic topic as *the study of non-linear systems of one degree of freedom and related oscillations.*

Given one of our systems, the most important physical question is that of its steady states. It may consist of a position of equilibrium or of an oscillatory motion. However, neither the one nor the other offers much physical interest unless it has a certain degree of permanency, i.e. unless it is stable. Herein lies the reason why the question of stability will be repeatedly discussed in the sequel.

Non-linear differential equations lend themselves but rarely to explicit computations. In spite of this we will frequently be in a position to say something about the general character of the solutions, and this information will often be most valuable. Thus while there is no convenient method for solving van der Pol's equation, it is known that: (a) there is a unique periodic solution and it is stable; (b) every solution tends asymptotically to the periodic solution. These two properties manifestly provide most valuable practical information.

CHAPTER I

Linear Systems

§1. Linear Systems without Friction

We initiate our study by considering a simple type of oscillatory system; one which is one-dimensional, linear, and conservative. Such an idealized system is approximated by a mass moving horizontally along a smooth bar under the action of two springs (Fig. 1). For small displacements the springs approximate an elastic restoring force, i.e. a force proportional to the displacement. We neglect air resistance and the internal friction of the springs. Under these assumptions the equation of motion for the idealized system is

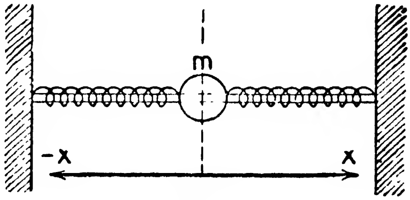

Fig. 1.

$$m\ddot{x} + kx = 0, \tag{1}$$

where x is the displacement from the equilibrium position and k is the spring constant ($k > 0$). Setting $k/m = \omega_0^2$, we obtain the differential equation of a *harmonic oscillator* in its so-called canonical form:

$$\ddot{x} + \omega_0^2 x = 0. \tag{2}$$

The corresponding electrical analogue is the simple, though again idealized, circuit with capacitance C and inductance L. No account is being taken of the energy dissipated in the circuit, the non-linearity of the elements, or the fact that the inductance and capacitance are distributed. In this case

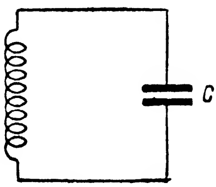

Fig. 2.

$$L\ddot{q} + \frac{q}{C} = 0$$

and with $1/LC = \omega_0^2$, $x = q =$ the charge on the capacitor, we have again equation (2). The results obtained for this idealized harmonic oscillator are limited in their application to small displacements (charges) and small velocities (currents), and to a description of the system during a limited number of oscillations. The limitation on the number of cycles is due to the dissipation in energy which has been neglected.

We now consider the characteristic properties of harmonic oscillators. The integral of (2) is of the form

$$x = A \cos \omega_0 t + B \sin \omega_0 t = K \cos (\omega_0 t + \alpha),$$
$$\dot{x} = -\omega_0 K \sin (\omega_0 t + \alpha).$$

If $x(0) = x_0$ and $\dot{x}(0) = \dot{x}_0$, then

$$\text{(3)} \qquad \begin{cases} K = \sqrt{A^2 + B^2} = \sqrt{x_0^2 + \dfrac{\dot{x}_0}{\omega_0^2}}, \\ \tan \alpha = -\dfrac{B}{A} = -\dfrac{\dot{x}_0}{\omega_0 x_0}. \end{cases}$$

We see that the displacement (charge) is represented by a "sinusoidal curve" (Fig. 3, drawn for the case $\alpha = -\pi/2$). This "sinusoidal curve" or harmonic oscillation is characterized by the three quantities:

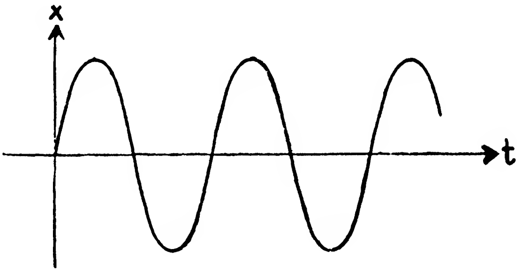

FIG. 3.

K, the *amplitude* or maximum deviation; ω_0, the *angular frequency* or number of oscillations during 2π seconds; and α *the phase angle.* The phase angle plays no important role if we are examining a single "isolated" process, since the initial time can always be so specified that $\alpha = 0$. It is of importance however when several processes are considered simultaneously. When $x_0 = 0$ and $\dot{x}_0 = 0$, i.e. when the oscillator is in equilibrium, $K = 0$ and it remains in equilibrium. Note also that the amplitude K is determined completely by the initial conditions, whereas the angular frequency ω_0 depends only upon the nature of the system. Thus (3) and the equation for ω_0 give an exact quantitative description of the motion of a harmonic oscillator.

§2. THE PHASE PLANE. APPLICATION TO HARMONIC OSCILLATORS

1. Phase space, phase plane. The general concept of a phase space is familiar in physics, and has been used extensively in kinetic theory of gases. Generally speaking a dynamical system with n degrees of freedom depends upon a certain number n of positional

coordinates $q_1, \cdots, q_n$ and the state of the system at time t is fixed by the values of the q_i and of their velocities $\dot{q}_i$. We may consider the $q_i,\dot{q}_i$ as coordinates of a space S of $2n$ dimensions called the *phase space* To each state of the system there corresponds the point M with coordinates $q_i,\dot{q}_i$ in the space S and as t varies the point M describes a curve called a *path,* which describes the history of the system. A familiar and closely related concept is the *world line* in the theory of relativity. The totality of all the paths may be described as the *phase portrait* of the system. It represents all the possible histories of the system; any one of them is determined by a single state M_0. Geometrically speaking this merely means that there is one and only one path through each point of the phase space S.

The phase portrait bears a close relation to the set of streamlines in what is known as a permanent (stationary) flow in hydrodynamics. For that reason it is often referred to as a *flow* in the literature, and the term is also extended to the dynamical system itself.

In our particular situation we are dealing with a system with *one* degree of freedom. It depends upon a single positional coordinate x and its state is determined at any time t by the values of x and of its velocity $\dot{x}$. The phase space will then be a phase plane. It is convenient to set $\dot{x} = y$, and so the state of the system is determined by the values of the coordinates x and y, where x is the positional coordinate and y the velocity of the system. The point $M(x,y)$ is referred to as the *representative point* of the system. Thus the phase plane represents the totality of all possible states of the given system. As t varies in the system, x,y will be functions of $t: x = f(t)$, $y = g(t)$, and M will describe a *path* π. The two equations just written are of the type known as *parametric equations.* The same equations represent also a *motion* on the path, that is to say a certain mode of describing the path as time varies. The velocity of the point M is called *phase velocity* and is not to be confused with the velocity of the system. The phase velocity is the plane vector whose components are $\dot{x},\dot{y}$, that is $\dot{x},\ddot{x}$, while the velocity of the system is merely the quantity $\dot{x}$. The complete path represents the actual history of the system throughout all time. The totality of all the paths represents all the *possible* histories. The particular path followed among all the possible paths is fixed when a single point is known, for instance the point $M_0(x_0,y_0)$, or in physical language when one state of the system is specified. This means here also that one and only one path goes through every point of the phase plane. The totality of all the paths constitutes the *phase portrait* of the system. Often by mere inspection of this portrait one may state

important properties of the system. This will be clearly shown throughout the sequel.

When one knows the solution of the differential equation of the harmonic oscillator (2), it is easy to find the equation of the path. Its parametric equations are

(4) $$x = K \cos (\omega_0 t + \alpha), \qquad y = -K\omega_0 \sin (\omega_0 t + \alpha).$$

Eliminating t we find for the path the equation

(5) $$\frac{x^2}{K^2} + \frac{y^2}{K^2\omega_0^2} = 1.$$

It is easy to see that this equation represents a family of similar ellipses. Through each point of the plane there passes one and only

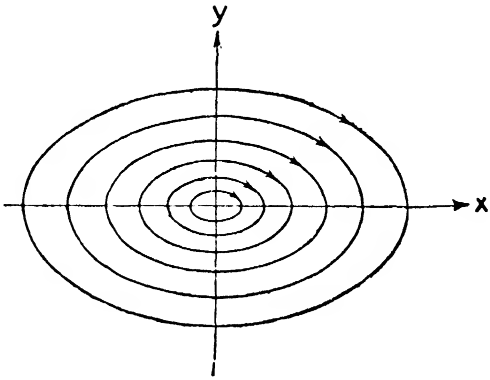

FIG. 4.

one ellipse; it corresponds to a given value of K, i.e. to a definite class of initial conditions, namely, to the same initial values of the total energy of the system. The whole x,y-plane, origin excepted, is covered with concentric ellipses. The ellipse "passing through" the origin is merely that point itself (Fig. 4).

Let us examine the motion of the representative point along one of the ellipses. With axes chosen as usual, the movement of the point will always be clockwise since a positive velocity ($y > 0$) corresponds to the increase of the value of x with time and a negative velocity ($y < 0$) corresponds to the decrease of x with time.

The phase velocity is a vector with components $\dot{x},\dot{y}$ in the x,y directions, respectively.

(6) $$v = (\dot{x},\dot{y}) = (-K\omega_0 \sin (\omega_0 t + \alpha), -K\omega_0^2 \cos (\omega_0 t + \alpha)).$$

It is tangent to the trajectory, points in the direction of motion, and

its length or magnitude is the phase speed

$$(7) \qquad |v| = K\omega_0 \sqrt{\sin^2(\omega_0 t + \alpha) + \omega_0^2 \cos^2(\omega_0 t + \alpha)}.$$

It is easily seen that $|v|$ remains between $K\omega_0$ and $K\omega_0^2$, and is not zero except for $K = 0$.

Thus the periodic motions of the system correspond on the phase plane to concentric closed paths on which the representative point is moving with a non-zero phase velocity and with the time for an entire revolution $T = 2\pi/\omega_0$. The equilibrium states of the oscillator correspond to a degenerate path consisting of a point.

In the case of the harmonic oscillator we have obtained the phase portrait by using the solution (4) in finite form of the equation of the oscillator. Instead of this, it is possible to derive conclusions concerning the motion of the representative point in the phase plane directly from (2) without having recourse to the explicit integrals (4) of the equation. It may therefore be applied in cases where one cannot obtain the integrals.

2. Equation not containing the time. To pass directly from the initial equation (2), without integrating it, to the representation on the phase plane, we shall proceed in the following way. We replace the initial equation of the second order by two equivalent equations of the first order:

$$(8) \qquad \frac{dx}{dt} = y, \qquad \frac{dy}{dt} = -\omega_0^2 x.$$

By dividing one of these equations by the other, we have the differential equation

$$(9) \qquad \frac{dy}{dx} = -\omega_0^2 \frac{x}{y}.$$

We can see that, if the dependence of x on t is expressed by a differential equation of the second order (2), the dependence of y on x is expressed by a differential equation of the first order. If we integrate (9) we will obtain the equation of the paths in finite form and not merely in differential form.

3. Singular points. A system such as (8) in which the time figures only as a differential dt is said to be *autonomous*. In such a system the points where both $\dot{x}$ and $\dot{y}$ vanish are called *singular points*. If $M(a,b)$ is such a point then $x = a$, $y = b$ is a solution of the system, that is to say M is a path reduced to a single point.

It is evident that $x = 0$, $y = 0$, i.e. the origin itself is the only

singular point of our system (8). At every non-singular point, (9) determines uniquely the slope of the path through the point.

An isolated point in whose neighborhood the paths consist of a set of concentric ovals (for example circles or ellipses) is known as a *center*. As (6) shows, the origin is a center for the system (8).

4. Isoclines. Equation (9) determines the "field" of tangents on the phase plane. Its nature is made clear by constructing a family of isoclines,[1] which in our case are straight lines passing through the

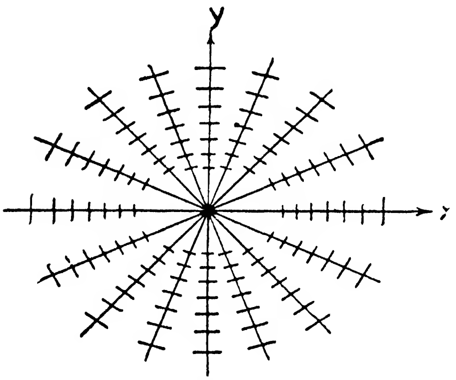

FIG. 5.

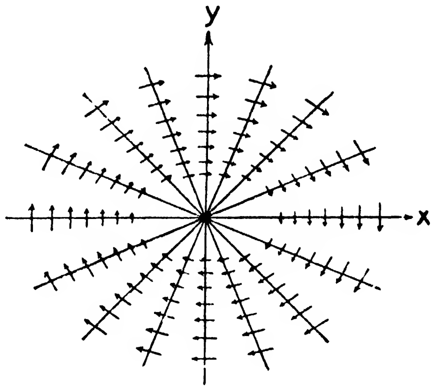

FIG. 6.

origin (Fig. 5). Let us determine the isocline corresponding to points where the slope of the path is β. Its equation is

$$\frac{dy}{dx} = -\omega_0^2 \frac{x}{y} = \beta; \qquad \text{or} \qquad y = \sigma x, \qquad \sigma = -\frac{\omega_0^2}{\beta}.$$

It is easy to see that the field under investigation consists of linear elements, symmetric with respect to the axes and changing gradually (with the change of the inclination σ) their direction from horizontal (along the y axis where $\beta = 0$) to vertical (along the x axis where $\beta = \infty$).

One cannot tell from (9) in what direction and with what velocity the representative point will move on the phase plane. Equation (8) defines the phase velocity in magnitude and direction; in fact its components are

$$v_x = \dot{x} = y, \qquad v_y = -\omega_0^2 x.$$

It is certainly useful to indicate the direction of motion as well as the slope of the path. This is done in Fig. 6.

As we have already indicated, the phase velocity, $v = \sqrt{y^2 + \omega_0^4 x^2}$, tends to zero only at the origin, i.e. at the singular point. From Figs.

[1] An isocline is a locus of points where the paths have a given slope.

5 and 6 it is easy to see that the isocline method enables us to obtain, in the case under consideration, some idea of the behavior of the trajectories on the phase plane. The isocline method offers little advantage whenever as in the present instance separation of variables makes it easy to integrate the equation. For we have at once from (9)

$$x\,dx + \frac{1}{\omega_0^2}\,y\,dy = 0, \tag{10}$$

whose integration yields

$$\frac{x^2}{2} + \frac{y^2}{2\omega_0^2} = C.$$

Setting $2C = K^2$, we have again the equation (5) of the family of ellipses on the phase plane. One should not forget that this time (5) has been derived in a completely different way, without knowledge of the solution of the differential equation (2). Whenever the equation, similar to (10), cannot be integrated, the isocline method may enable one to obtain nevertheless a fairly accurate idea of the behavior of the paths.

5. Equilibrium and periodic motion. Passing to a converse problem, what can we say about the character of the motion when we know the path and the phase velocity?

Here all paths other than the origin correspond to periodic motions. They are all ellipses, i.e. closed curves. When our representative point describes a closed curve once, so that the system acquires, after a certain time, the same position and the same velocity, then its movement beyond will coincide exactly with the previous one; the process will repeat itself.

We see that the "time of return," i.e. the period, is finite. In fact, the length of our ellipse is finite; the phase velocity along the ellipse never approaches zero; it is equal to zero only at the origin, but our ellipses do not pass through the origin. Hence the representative point describes the ellipse in a finite time. Furthermore the degenerate path $x = 0$, $y = 0$ corresponds to the state of equilibrium. The phase velocity at the point $x = 0$, $y = 0$ is zero; the representative point, situated at the initial moment at the origin, will remain there if accidental displacements or shocks do not throw it out of this position.

The singular points, those for which $\dot{x} = 0$, $\dot{y} = 0$, correspond to states of equilibrium. This is understandable also from the physical point of view. For example, in mechanical systems, the condition

$\dot{x} = 0$ indicates that the velocity is zero, while $\dot{y} = 0$ shows that the acceleration or force, which is the same thing, is zero.

In general, for dynamical systems the converse of the above statement is also true: the equilibrium states correspond to singular points.

Thus without knowing the possible movements quantitatively we do know their qualitative characteristics. This qualitative analysis of a linear frictionless system (harmonic oscillator) may be formulated in the following way: *Under any initial conditions the system undergoes a periodic motion around the equilibrium state $x = 0$, $y = 0$, with the exception of the case when the initial conditions correspond to equilibrium.*

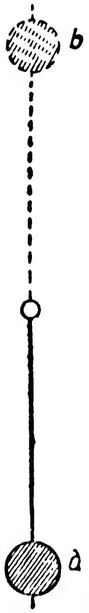

FIG. 7.

§3. STABILITY OF EQUILIBRIUM

We can visualize intuitively the meaning of the statement "stability of equilibrium." The intuitive notion is of course insufficient, and it is necessary to transform it into an exact and useful notion.

Consider the simple example of a mathematical frictionless pendulum (Fig. 7). Two states of equilibrium of the pendulum are possible:

1. When we place it at the lowest point a, without initial velocity.
2. When we place it at the highest point b, without initial velocity.

The lower state of equilibrium is stable while the upper is unstable. In fact, if the pendulum is in b, the slightest impulse makes it move with an accelerating velocity, and it leaves the immediate vicinity of the point b. At the point a the pendulum will behave quite differently: after a slight impulse the pendulum begins to move with decreasing velocity; the smaller the impulse, the smaller the displacement from a before it turns back and starts to oscillate about a. If the impulse is sufficiently small, the pendulum will remain in a certain region around a, and its velocity will not exceed a certain value.

With this example as a guide we shall endeavor to give a definition of stability and use the phase plane for the purpose. Let us examine a system in a state of equilibrium. The representative point in the phase plane is at rest at one of the singular points. If we disturb the equilibrium by an impulse,[1] the representative point will leave the

[1] Generally in connection with stability one considers instantaneous impulses, their role being to impress an instantaneous displacement upon the representative

singular point and will begin to move over the phase plane. Let us draw the representative point in black and the singular point in white (Fig. 8). We can then characterize stable equilibrium in the following way: If for a sufficiently small initial displacement of the black point, it remains near the white point, then the white point represents a stable state of equilibrium.[1]

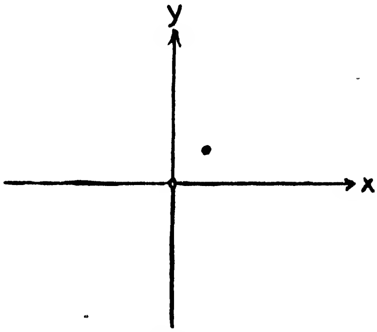

Fig. 8.

It is clear that this characterization is also incomplete. First of all, shall we call the white point stable if the black point does not move far away when initially displaced in one direction but moves far away for the smallest displacement in another direction? Obviously the white point will then be unstable; it will be, so to speak, "conditionally" stable if we eliminate a certain class of displacements. It is necessary to require of the black point that it does not go far away from the white point as a result of a small displacement in any direction.

Secondly, and this is more important, the expressions "does not go far away" and "remains in the vicinity" etc. are not sufficiently precise. The notions "near" and "far" depend on the physical conditions of the problem. We shall therefore formulate the following definition:

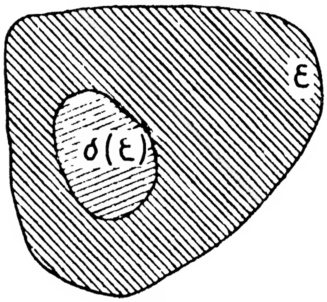

Fig. 9.

A state of equilibrium is stable whenever given any region ϵ containing it there is another $\delta(\epsilon)$ in ϵ such that any motion starting in the region δ remains in the region ϵ. The state of equilibrium is unstable whenever given any region ϵ containing it each region in ϵ containing it contains a point M such that a motion starting at M leaves ϵ.

While these definitions are expressed in terms of the phase plane, they may be formulated without having recourse to the phase plane. One can, in fact, translate the definition of stability into the language of mathematical inequalities, defining the movement of the black point

point in the phase plane or, equivalently, an instantaneous modification of the initial conditions. This is, of course, an idealization of true impulsions.

[1] Frequently this is also formulated as follows: The equilibrium is stable whenever a small disturbance remains small.

after the disturbance by $x(t), y(t)$, and assuming, for the sake of simplicity, that the region of admissible deviations ϵ represents a square (Fig. 10). The above definition may then be stated as follows: The state of equilibrium $x = \bar{x}$, $\bar{y} = 0$ is stable whenever for any small $\epsilon > 0$ we can find a $\delta(\epsilon) > 0$ such that, if at time $t = 0$ we have $|x(0) - \bar{x}| < \delta$ and $|y(0)| < \delta$, then for $0 < t < +\infty$ we have $|x(t) - \bar{x}| < \epsilon$ and $|y(t)| < \epsilon$. We shall refer to this stability as "stability according to Liapounoff" and always have it in mind when we speak of stability. Later we shall discuss other definitions of stability and realize the significance of the work of Liapounoff [3] on stability.

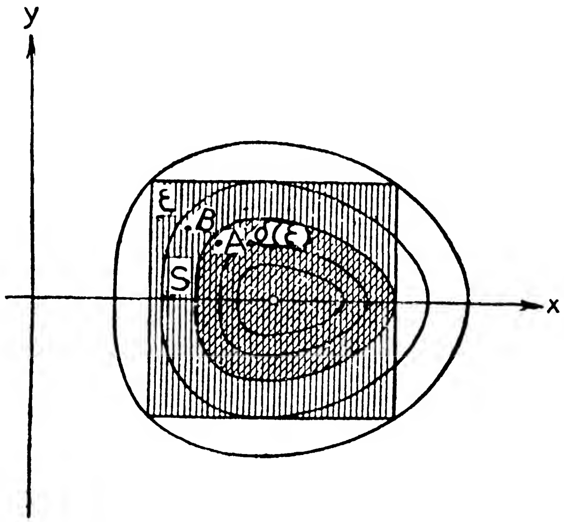

FIG. 10.

We now take up the stability of the state of equilibrium of a harmonic oscillator. This investigation will show why one has to consider two regions ϵ and δ in the definition of stability.

It is rather easy to show that the center type of singular point corresponds to stable equilibrium. Let the given region be any small region ϵ, for example a square (region shaded by vertical lines in Fig. 10). Let us choose among the closed paths surrounding the singular point a curve S, tangent to the given square and interior to it. That is always possible, independently of whether the closed paths in the immediate vicinity of the point have the form of ellipses or any other form. The existence of the curve in question requires only the continuity of the system of concentric ovals, around a point, a situation we always meet with in the case of a center. The region inside the curves (shaded with cross lines) will represent the region $\delta(\epsilon)$ for, if

the initial position of the black point is inside this region (point A), then it will never leave the square ϵ but will undergo periodic motion around the state of equilibrium. We could, of course, choose for the region δ any other region situated inside the curve S, for example, the region inside the square having all its points inside the curve S.[1] Thus we may affirm that *the state of equilibrium of the center type is stable.*

§4. Linear Oscillator with Friction

In the remainder of the chapter we shall discuss non-conservative systems, and assume linear friction, i.e. a frictional force proportional to the velocity $\dot{x}$. (This is called viscous damping and is closely approximated for small velocities in air and in liquids.) The equation of motion is still linear and is

$$m\ddot{x} + b\dot{x} + kx = 0 \tag{11}$$

where b is the friction coefficient, i.e. the frictional force per unit of velocity. An electrical example analogous to the mechanical system with friction proportional to velocity is an RLC circuit with linear resistance. Such a circuit obeys the equation

$$L\ddot{q} + R\dot{q} + \frac{q}{C} = 0 \tag{12}$$

where q is the charge of the condenser, and L, R, and C, as usual, inductance, resistance, and capacitance.

If $b/m = 2h$, $k/m = \omega_0^2$ [or, correspondingly $R/L = 2h$, $1/LC = \omega_0^2$], we reduce (11) and (12) to the usual form:

$$\ddot{x} + 2h\dot{x} + \omega_0^2 x = 0. \tag{13}$$

The general solution of this equation is

$$x = Ae^{\lambda_1 t} + Be^{\lambda_2 t} \tag{14}$$

where λ_1 and λ_2 are the roots of the characteristic equation

$$\lambda^2 + 2h\lambda + \omega_0^2 = 0.$$

They are real for $h^2 \geqslant \omega_0^2$ and complex for $h^2 < \omega_0^2$.

According to the sign of $h^2 - \omega_0^2$, we obtain two types of solutions and consequently two processes. For $h^2 < \omega_0^2$ we have a *damped oscillatory process*, and for $h^2 > \omega_0^2$ a *damped aperiodic process* (subsidence).

[1] Clearly one cannot generally take ϵ itself as $\delta(\epsilon)$, for if we take, say, B as the initial position in ϵ (Fig. 18), the path will leave ϵ.

The constants in equations (11), (12) or (13) are sometimes referred to as *oscillatory parameters.*

1. Damped oscillatory process. Setting $\omega_1^2 = \omega_0^2 - h^2$ the general solution (14) becomes

$$x = e^{-ht}(A \cos \omega_1 t + B \sin \omega_1 t) = Ke^{-ht} \cos (\omega_1 t + \alpha). \tag{15}$$

If at $t = 0$, $x = x_0$ and $\dot{x} = \dot{x}_0$ the constants in (15) are $A = x_0$, $B = \dfrac{\dot{x}_0 + hx_0}{\omega_1}$, and $K^2 = A^2 + B^2$, $\tan \alpha = -\dfrac{B}{A}$. The velocity is

$$\dot{x} = -Ke^{-ht}(h \cos (\omega_1 t + \alpha) + \omega_1 \sin (\omega_1 t + \alpha)). \tag{16}$$

Formulas (15) and (16) define an exponentially damped oscillation.

The functions $x(t)$ and $\dot{x}(t)$ which we obtained *are not periodic functions.* In fact a function $f(t)$ is said to be periodic when and only when there is a non-zero number τ such that

$$f(t + \tau) = f(t)$$

for every value of the argument t. The lowest positive such τ is called the *period* of the function. The functions (16) and (17) are not periodic. Strictly speaking they have no period. However the time between consecutive zeros of x (or zeros of $\dot{x}$) is $T_1/2 = \pi/\omega_1$. We call $T_1 = 2\pi/\omega_1$ the conditional period of the damped oscillation.

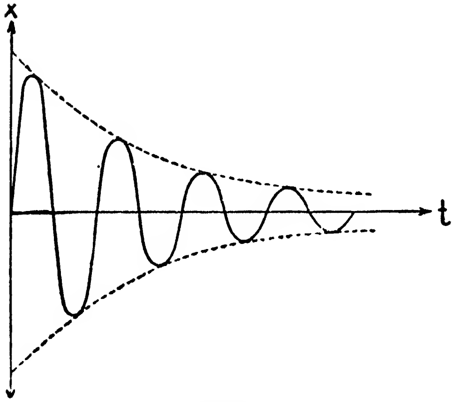

FIG. 11.

The graph of $x(t)$ is shown in Fig. 11.[1] The rate of damping can be characterized by h, *the damping coefficient.* The time for x to decrease by a factor $1/e$ is $1/h$, *the time constant.* (e is the base of the natural logarithm and $1/e$ represents a percentage decrease of about 63 per cent.) The disadvantage of h is that its value depends upon the units for time. For this reason, we introduce a dimensionless measure of the damping, the *logarithmic decrement* $d = T_1 h$. From (15)

$$\log \frac{x(t)}{x(t + T_1)} = d = T_1 h.$$

[1] Notice that both maxima and minima are not half-way between the appropriate zeros but shifted to the left of that position.

Hence we see that the logarithmic decrement is the logarithm of the ratio of displacements one period apart. To add further to the significance of d, we note that $1/d$ is the number of conditional periods after which the amplitude decreases to $1/e$ of its value.

It is easy to see that the law obtained for the damping of the oscillations is closely related to the idealization of the law of friction. It is only when one assumes linear friction that one obtains the law of decrease of maxima according to a geometric progression with e^{-d} as ratio. On the other hand, it is clear that the notion of logarithmic decrement loses its meaning if the damping law is such that the ratio between two consecutive maxima does not remain constant. Consequently logarithmic decrements occur only in linear systems.

The definition of the logarithmic decrement can be deduced from the curve represented in Fig. 11, or, more conveniently, from the relation between the maxima (or minima) and time represented on a semi-logarithmic scale (instead of deviations one uses logarithms at maxima as ordinates). In the latter case this dependence is represented by a straight line, the slope of which is the damping coefficient h; multiplying it by the conditional period T_1 we obtain the logarithmic decrement d.

Thus we can characterize a damped oscillation by four quantities: the conditional period T_1 (or the corresponding angular conditional frequency ω_1), the logarithmic decrement d, the conditional amplitude K, and the phase angle α.

The conditional period and the logarithmic decrement are determined by the properties of the system. For a given oscillatory mechanism, the amplitude and the phase remain undetermined for they are defined by the initial conditions.

2. Linear transformations of the plane. Linear transformations are of sufficient importance to make it worth while to present a geometric picture of their effect. A (non-singular) *linear transformation* is defined by

$$\left\{\begin{aligned} u &= ax + by, \\ v &= cx + dy, \end{aligned}\right. \qquad \Delta = ad - bc \neq 0. \tag{17}$$

The transformation (17) can be considered from two points of view, either in a "passive" sense or in an "active" sense, i.e. either as a transformation from rectangular coordinates x,y to non-rectangular coordinates u,v conserving geometric figures (the usual "passive" interpretation of analytical geometry), or as a deformation of the figures without changing the coordinate system ("active" interpretation of such transformations, such as occur for example in elasticity).

Since $\Delta \neq 0$, no pair of distinct points is mapped by (17) into the same point, and the transformation is said to be *non-singular*. The transformation is called *linear* because straight lines go into straight lines. By applying (17), we see that the line

$$x = \alpha y + \beta \tag{18}$$

is transformed into the line

$$u = \alpha' v + \beta' \tag{19}$$

where

$$\alpha' = \frac{b + \alpha a}{d + \alpha c}, \qquad \beta' = \frac{\beta \Delta}{d + \alpha c}$$

Observe that $d + \alpha c = 0$ is a special case and corresponds to v = constant.

We can now draw the following conclusions. Straight lines go into straight lines. If $\beta = 0$ then $\beta' = 0$, and lines through the origin go into lines through the origin. By noting that α' does not depend on β, we see that a family of parallel lines goes into a family of parallel lines. The distances of (18) and (19) from the origin are $d = \beta/\sqrt{1 + \alpha^2}$ and $d' = \beta'/\sqrt{1 + \alpha'^2}$ and d/d' depends only upon α. Thus a family of equally-spaced parallel lines goes into a family of equally-spaced parallel lines. Thus under a linear transformation lines may be rotated (and reflected) and they may be stretched (or compressed), but in any one direction they are stretched uniformly. One can obtain a good picture of this by considering the transformation of a grid made up of two families of equally-spaced parallel lines. This grid will then map into a grid made up of two families of equally-spaced parallel lines, though the grid may be rotated about the origin and the sides of the grid may be stretched or compressed. Often it is possible to find a grid which is mapped into a similar grid (the angle between the lines is left unchanged), though this is not always the case.

3. Representation of damped oscillations on the phase plane. Equations (15) and (16) are the parametric equations of the family of paths. Thus

$$\begin{cases} x = Ke^{-ht} \cos (\omega_1 t + \alpha) \\ y = \dot{x} = -Ke^{-ht}(h \cos (\omega_1 t + \alpha) + \omega_1 \sin (\omega_1 t + \alpha)). \end{cases} \tag{20}$$

We shall show that these equations represent a family of spirals with an asymptotic point at the origin. To do so we shall make a linear

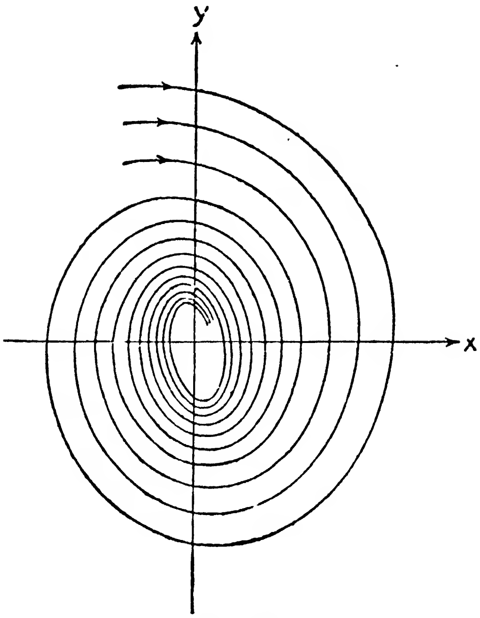

FIG. 12.

transformation of coordinates, a method we shall frequently apply in the sequel. Introduce the new variables

$$u = \omega_1 x, \qquad v = hx + y. \tag{21}$$

The system (20) becomes then

$$\begin{cases} u = \omega_1 K e^{-ht} \cos(\omega_1 t + \alpha) \\ v = -\omega_1 K e^{-ht} \sin(\omega_1 t + \alpha) \end{cases} \tag{22}$$

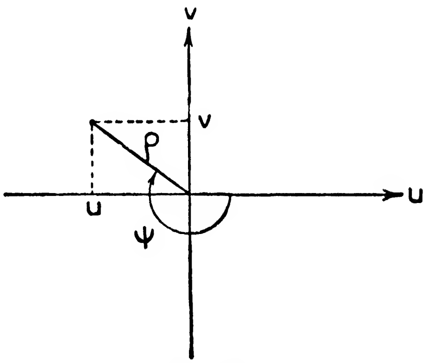

FIG. 13.

or in polar coordinates (Fig. 13) in the u,v-plane

$$\rho = \omega_1 K e^{-ht} = ce^{-\frac{h}{\omega_1}\psi}, \tag{23}$$

where $\rho = \sqrt{u^2 + v^2}$ and $\psi = \omega_1 t$. As t increases the point on the curve moves clockwise with angular velocity ω_1 and the radius vector decreases exponentially. The curve has an asymptotic point at the origin and is known as a *logarithmic spiral.*

Equation (21) represents a linear transformation of the phase plane and from our study of the linear transformations the picture of the paths in the x,y-plane can be made clear. By this simple linear trans-

formation the grid formed by the lines parallel to the y-axis and the lines whose slopes are $h/(\omega_1 - 1)$ is deformed into a similar grid (the angles are preserved) without rotation. In the "active" sense the deformation is produced by stretching the lines whose slope is $h/(\omega_1 - 1)$ by a factor ω_1. The lines parallel to the y-axis are not stretched. In the case $\omega_1 = 1$, the points on the line $x = c$ are displaced a distance hc.

Thus under the reversal of this deformation the family of logarithmic spirals in the u,v-plane will go into a family of spirals which wind toward the origin in a clockwise direction. Since lines through the origin go into lines through the origin, the turning of the radius vector to the path with increasing time has the same conditional period $T_1 = 2\pi/\omega_1$ in both planes. The angular velocity of turning is constant in the u,v-plane and in general not constant in the phase plane. As we know, the radius vector is stretched uniformly and hence the logarithmic decrement of the decrease of its length, the same in both planes, is $d = hT_1$. The family of logarithmic spirals covers the u,v-plane and one and only one spiral passes through each point. Clearly this must also be true of the family of spiral paths in the phase plane. The origin $x = 0$, $y = 0$ is again an exception. It is a degenerate path at the origin. The phase velocity is zero at this point and does not vanish elsewhere. It does decrease with each revolution and can be shown to have the logarithmic decrement $d = hT_1$.

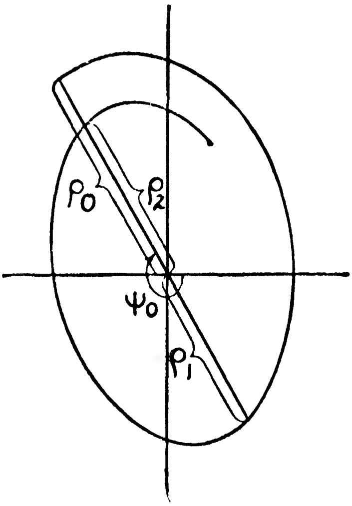

FIG. 14.

4. Direct investigation of the differential equation. The preceding results may also be derived directly from (13) without referring to the solution (14). To that end, replace the initial equation of second order (13) by two equivalent equations of first order:

$$\frac{dx}{dt} = y, \qquad \frac{dy}{dt} = -2hy - \omega_0^2 x, \tag{24}$$

and hence

$$\frac{dy}{dx} = -\frac{2hy + \omega_0^2 x}{y}. \tag{25}$$

We can see immediately that this equation, like (9), determines a

certain field of tangents on the phase plane, and together with (24) it will determine a vector field having just one singular point $x = 0$, $y = 0$. By means of isoclines we can easily make an approximate study of the character of this field. If the paths are to have the slope χ, the equation of the corresponding isocline will be

$$-\frac{2hy + \omega_0^2 x}{y} = \chi, \qquad \text{or} \qquad y = \sigma x$$

where

$$\sigma = -\frac{\omega_0^2}{\chi + 2h}, \tag{26}$$

i.e. the isoclines will be straight lines through the origin. If we give to χ a sufficiently large number of values (h and ω_0 being fixed by the system), we obtain a family of isoclines which will enable us to construct a field with a suitable degree of precision. Fig. 15 represents such a field constructed by means of a few isoclines. This figure enables one to visualize the nature of the paths. One may also easily integrate (25) by a well-known procedure for homogeneous equations. We thus obtain here ($\omega_0^2 > h^2$) for the paths the equation

$$y^2 + 2hxy + \omega_0^2 x = Ce^{2\frac{h}{\omega_1} \operatorname{arctg} \frac{y + hx}{\omega_1 x}}$$

derived without knowing the solution of (13). The expression for the phase velocity v can be deduced from (24):

$$v^2 = \omega_0^4 x^2 + 4h\omega_0^2 xy + (1 + 4h^2)y^2.$$

By this method we can see at once that the phase velocity only becomes zero at the origin. It decreases steadily as the representative point approaches the origin.

Given the character of the paths and the expression of the phase velocity, what can we say regarding the motions? First, we may affirm that all the paths (other than $x = 0$, $y = 0$) correspond to oscillatory damped motions tending to the state of equilibrium. In fact, all the paths are spirals; when the representative point moves on one of the spirals, the coordinate and the velocity pass many times through zero and therefore the spirals on the phase plane correspond to an oscillatory process. Furthermore, the radius-vector of the representative point moving on a spiral decreases with each revolution. This means that we are dealing with a damped process. The maximum values of x and $\dot{x}$ decrease from one revolution to the next. It is also clear that the origin corresponds to the state of equilibrium.

To sum up then: *Whatever the initial conditions the system undergoes a damped oscillatory motion around the state of equilibrium* $x = 0$, $y = 0$ *with the exception of the case when the initial conditions coincide exactly with the state of equilibrium.*

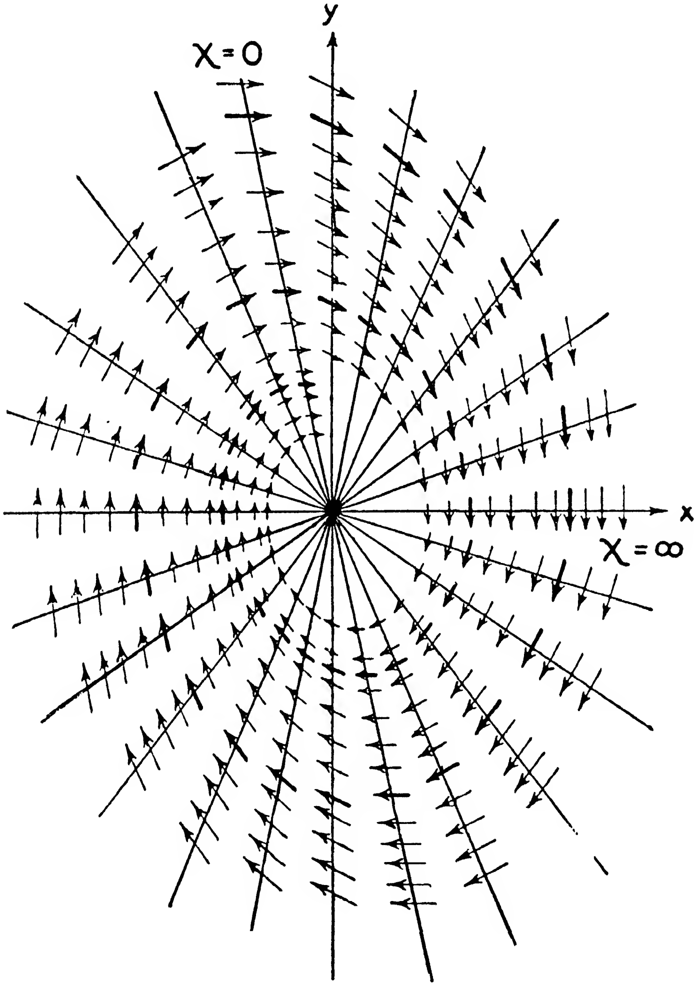

FIG. 15.

A singular point such as above, which is the common asymptotic point of a family of paths consisting of concentric spirals, is called a *focus*.

Let us now examine whether the focus is stable. Since along any path the representative point will move towards the singular point, it becomes clear that the condition of stability for the equilibrium state, formulated above, is fulfilled. In fact, we can always choose such a

region δ (doubly shaded in Fig. 16) that the representative point does not cross the boundary of the region ϵ (singly shaded in Fig. 16). Consequently the equilibrium is stable and the singular point is a stable focus. The stability of the focal type of singular point is apparently related to the fact that the paths are winding or unwinding with respect to the movement of the representative point. Since the direction of the movement of the representative point is in a sense determined by the choice of coordinates (the point must move clockwise), the stability of the singular point is also determined with a sense (since the direction in which the time is measured cannot be changed). Conversely, if the spirals should unwind, the singular point would be unstable. Equation (23), for example, shows that the winding of the paths is conditioned by $h > 0$, since it is only in this case that the radius-vector decreases for a clockwise motion (obviously we consider ω_1 positive and real). Thus, generally speaking, a focus can be either stable or unstable, while the center type of singular point, as we have seen, is always stable. In the present case, the focus is stable because $h > 0$. The physical meaning of this condition of stability is clear; friction must be positive, i.e. it must resist the motion and dissipate energy. Such positive friction resists motion; its overcoming requires work to be spent, and it cannot induce instability. If the equilibrium of the system is already stable in the absence of friction (in the harmonic oscillator), then it will remain stable in the presence of positive friction. In the sequel we shall also consider unstable foci.

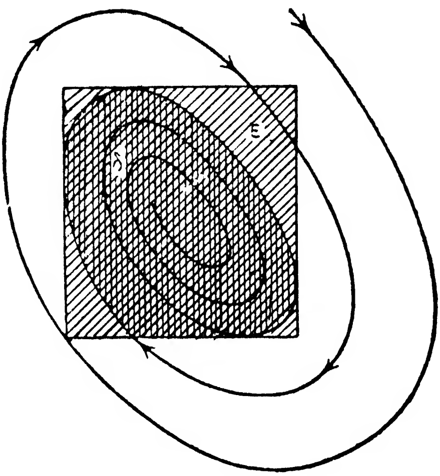

FIG. 16.

The stable focus possesses a "stronger" stability than the center examined in the previous paragraph. Actually, in the case of the stable focus, not only the condition of stability according to Liapounoff but even a stronger condition will be fulfilled. Namely, for all initial conditions, the system, after a sufficiently long period of time, will return as close as one wishes to the state of equilibrium. Such a stability, for which the initial deviations do not increase but on the contrary dampen, will be called *asymptotic stability.* In the case of the linear oscillator that we have studied, the focus is asymptotically stable.

5. Damped aperiodic process. Let us now examine the case when the roots of the characteristic equation are real, i.e. when $h^2 > \omega_0^2$. Setting $q^2 = h^2 - \omega_0^2$ we obtain the solution in the form:

$$x = e^{-ht}(Ae^{qt} + Be^{-qt})$$

or, introducing the notations

$$\lambda_1 = -h + q = -q_1; \qquad \lambda_2 = -h - q = -q_2$$

(so that $q_2 > q_1 > 0$), in the form

$$x = Ae^{-q_1t} + Be^{-q_2t}.$$

Here A and B are determined by the initial conditions. Namely, if for $t = 0$, $x = x_0$, and $\dot{x} = \dot{x}_0$, then

$$x = \frac{\dot{x}_0 + q_2x_0}{q_2 - q_1}e^{-q_1t} + \frac{\dot{x}_0 + q_1x_0}{q_1 - q_2}e^{-q_2t}. \tag{27}$$

It is clear that whatever the initial condition the movement is damped, since $q_1 > 0$ and $q_2 > 0$, and therefore, for $t \to +\infty$, $x(t) \to 0$. In order to learn more about the character of the damping, we have to find that t_1 and t_2—the times (i.e. the intervals of time after the initial moment) for which x and $\dot{x}$ respectively become equal to zero. From (27) we deduce the following equations for t_1 and t_2:

$$e^{(q_2-q_1)t_1} = \frac{\dot{x}_0 + q_1x_0}{\dot{x}_0 + q_2x_0} = 1 - \frac{x_0(q_2 - q_1)}{\dot{x}_0 + q_2x_0} \tag{28}$$

$$e^{(q_2-q_1)t_2} = \frac{q_2(\dot{x}_0 + q_1x_0)}{q_1(\dot{x}_0 + q_2x_0)} = 1 + \frac{\dot{x}_0(q_2 - q_1)}{q_1(\dot{x}_0 + q_2x_0)}. \tag{29}$$

It is easy to see that each of these equations has only one root, so that a damped oscillating process cannot occur—we are dealing with a so-called aperiodic process. Let us examine the situation when the equation (29) for t_2 has no positive roots. In that case the damping of the motion is monotonic, tending asymptotically to zero. This will take place when in the expression (29) for t_2

$$\frac{\dot{x}_0}{\dot{x}_0 + q_2x_0} < 0.$$

The region of initial values satisfying this inequality (region II) is represented in Fig. 17. For all other initial values

$$\frac{\dot{x}_0}{\dot{x}_0 + q_2x_0} > 0$$

and the equation defining t_2 has positive roots. It means that the displacement does not decrease monotonically, but its absolute value first increases and then, only after having reached its extreme position, does it start to decrease, tending asymptotically to zero.

Here it is necessary to distinguish two cases according to whether under the initial conditions the equation determining t_1 does or does not

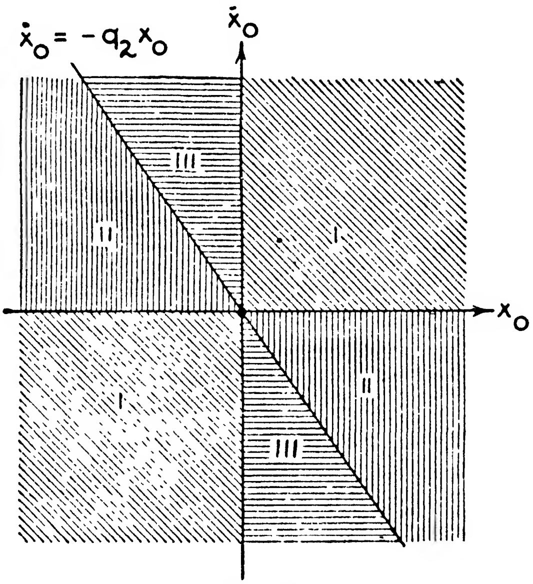

FIG. 17.

have a positive root. If there is no positive root, then the displacement during the entire motion $(0 < t < +\infty)$ retains its sign; the system leaves the state of equilibrium, reaches a certain maximum deviation and then approaches monotonically the position of equilibrium, but does not go through it. According to (28) this takes place when

$$\frac{x_0}{\dot{x}_0 + q_2 x_0} > 0.$$

The regions I of Fig. 17 correspond to this situation.

If the equation determining t_1 has a positive root, the system will first approach the position of equilibrium; at $t = t_1$ it will go through the position of equilibrium; later[1] at $t = t_2$ it will reach a certain

[1] From (28) and (29) we deduce the relation

$$e^{(q_2-q_1)(t_2-t_1)} = \frac{q_2}{q_1} > 1.$$

Hence the exponent at the left is positive and, since $q_2 - q_1 > 0$, also $t_2 - t_1 > 0$, or t_2 is a later time than t_1.

maximum deviation in the direction opposite to the initial deviation, and will finally approach monotonically the equilibrium position, without ever reaching it. This corresponds to the regions III in Fig. 17.

The relationship between the character of the motion and the initial conditions can be exhibited graphically in another way, namely

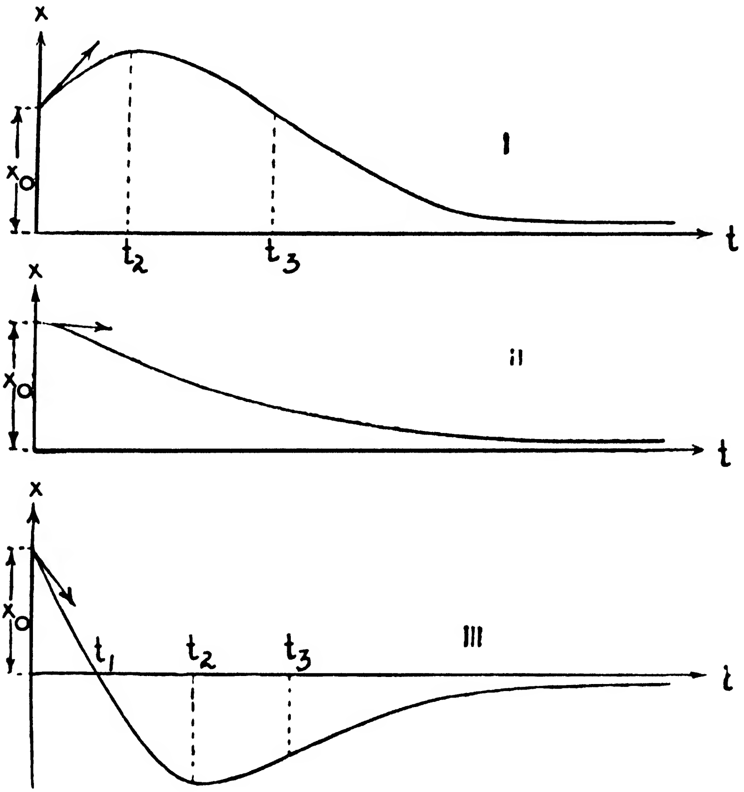

FIG. 18.

by representing the dependence of the displacement on the time: the graph $x(t)$, for the three cases, I, II, and III. This is done in Fig. 18, where we assume that the initial displacement $x_0 > 0$.

6. Representation of aperiodic processes on the phase plane. We have found the general solution

$$x = Ae^{-q_1 t} + Be^{-q_2 t},\ \dot{x} = y = -Aq_1 e^{-q_1 t} - Bq_2 e^{-q_2 t},$$

from which follows by a simple calculation

$$(y + q_1 x)^{q_1} = C_1(y + q_2 x)^{q_2}. \tag{30}$$

Since $q_1 \neq q_2$ it is no restriction to suppose that $q_1 < q_2$. To analyze

the family of curves (30) we shall use again a linear transformation of coordinates

$$y + q_1x = v, \qquad y + q_2x = u.$$

After this transformation, (30) will have, in terms of the new variables, the simple form:

$$v = Cu^a, \qquad a = \frac{q_2}{q_1} > 1.$$

Let u and v be rectangular coordinates. We can say that after the transformation we shall obtain the family of "parabolas" whose type is determined by the value of the exponent $a = q_2/q_1$. Independently of the character of the exponent, however, (i.e. whether it is an integer or a fraction, odd or even, etc.) we may assert the following:

1. All the paths, with the exception of the curve corresponding to $C = \infty$, are tangent at the origin to the u axis. For $dv/du = Cau^{a-1}$ and hence $(dv/du)_{u=0} = 0$.

2. The paths degenerate into straight lines for $C = 0$ and $C = \infty$. When $C = 0$ we have $v = 0$, i.e. the u axis; when $C = \infty$ we have $u = 0$, i.e., the v axis.

3. The paths turn their convexity towards the u axis (since v increases faster than u), and the absolute value of the ordinate increases monotonically with increasing u.

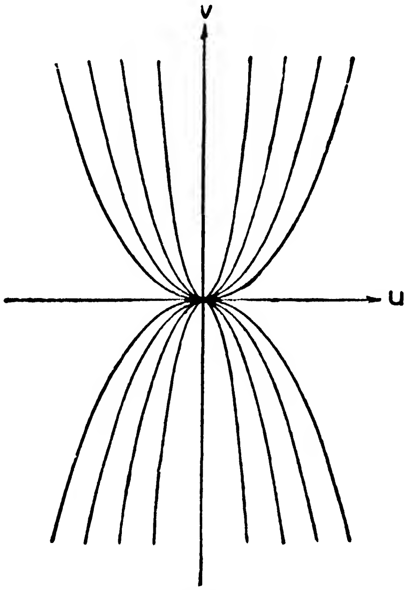

FIG. 19.

A family of such parabolas is represented in Fig. 19.

Let us return to the x,y plane. To the v axis on the u,v plane there corresponds the straight line $y + q_2x = 0$; to the u axis there corresponds the straight line $y + q_1x = 0$. The other paths or, more precisely, the other curves of the family (30) on the x,y plane represent deformed parabolas tangent to the straight line $y = -q_1x$. In order to draw this family of curves it is convenient to take into account the following properties which are readily deduced from (30):

1. The curves of the family have unlimited parabolic branches parallel to the straight line $y = -q_2x$.

2. They have tangents parallel to the x axis at the points of intersection with the line

$$y = -\frac{q_1 q_2}{q_1 + q_2} x, \qquad \left(\frac{q_1 q_2}{q_1 + q_2} < q_1\right).$$

3. They have tangents parallel to the y axis at the points of intersection with the x axis.

This family of curves is represented in Fig. 20.

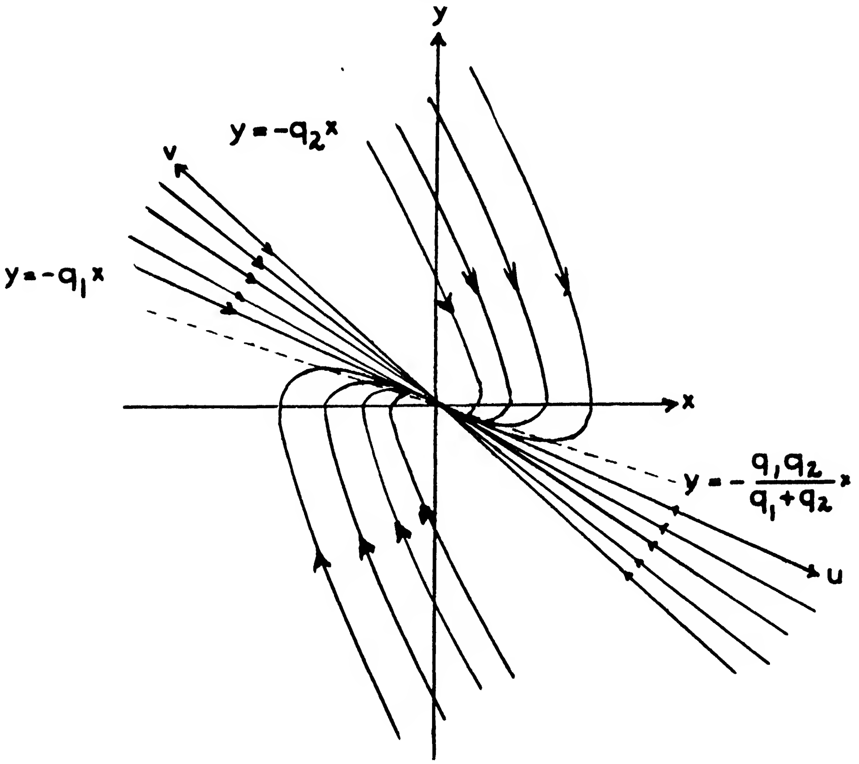

FIG. 20.

The same results may be derived without integrating the differential equation (13). If we replace this equation of the second order by two equivalent equations of the first order and eliminate t, we obtain the same equation for the paths as in the previous case:

$$\frac{dy}{dx} = -\frac{2hy + \omega_0^2 x}{y}. \tag{25}$$

The only singular point of this family of curves is again the origin, and it corresponds here also to the state of equilibrium. The isoclines

will now be straight lines defined by (26). Since here $h^2 > \omega_0^2$, the situation of the isoclines will be somewhat different (Fig. 21). If we set $z = y/x$ as in the integration of (25), we obtain (owing to $h^2 > \omega_0^2$) a result different from the previous cases, namely, the equation of the family of paths is of the "parabolic type":

$$y^2 + 2hxy + \omega_0^2 x^2 = C\left(\frac{\frac{y}{x} + h - \sqrt{h^2 - \omega_0^2}}{\frac{y}{x} + h + \sqrt{h^2 - \omega_0^2}}\right)^{\frac{h}{\sqrt{h^2-\omega_0^2}}}$$

or

$$(30) \qquad (y + q_1 x)^{q_1} = C_1(y + q_2 x)^{q_2}$$

where $q_1 = h - \sqrt{h^2 - \omega_0^2}$ and $q_2 = h + \sqrt{h^2 - \omega_0^2}$, i.e. the same equation as we obtained above by a different method (eliminating t from the solution of the initial differential equation).

The direction of the movement of the representative point is determined by the same reasoning as in the previous case, namely, from the condition that for $y = \dot{x} > 0$ the value of x increases. Since the slope of the tangent to the path changes its sign only once when crossing the x axis, it becomes evident that the representative point will describe the paths in the direction of the arrows on Fig. 20, i.e. it will always approach the origin. The velocity of the motion of the representative point becomes zero, as in the preceding cases, only when $\dot{x} = 0$ and $\dot{y} = 0$ simultaneously, i.e. at the singular point of the differential equation.

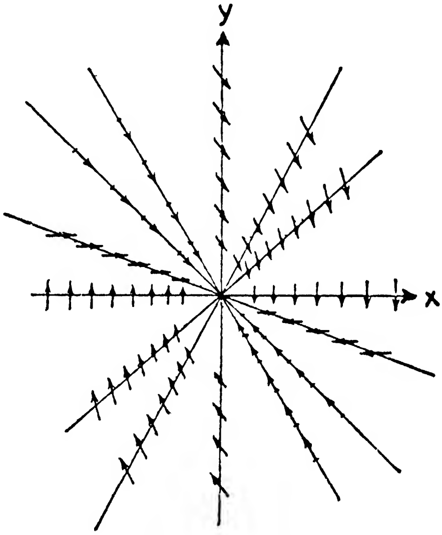

FIG. 21.

A point such as the origin to which all the paths converge is known as a *node*. It corresponds here to equilibrium, and it is easily seen that this equilibrium is stable, since the representative point moves along the paths towards the node. For this reason the node under consideration is said to be *stable*. The same remark applies as to the stable focus: If in a frictionless system the singular point (a center) is already stable, the addition of friction must necessarily preserve stability.

Let us examine in more detail the physical aspects of the three types of aperiodic motion represented in Fig. 18. First of all, if the initial velocity and the initial deviation have the same sign (i.e. if the representative point lies in the region I of Fig. 17), the system will first move from the position of equilibrium with a progressively decreasing velocity (the initial kinetic energy is used to increase the potential energy and to overcome friction). When the velocity reaches zero (point t_2) the system starts to return to the equilibrium position; the velocity will first increase since the *restoring force* decreases (for the system approaches equilibrium), and consequently beginning with a certain moment (point t_2 on Fig. 18, I) the velocity which is then maximum starts to decrease. The system will approach asymptotically the position of equilibrium.

Another case where the initial velocity and the initial deviation have different signs, i.e. the initial impulsion is directed in the sense opposite to the initial deviation, will lead to two different types of motion (II and III). If the initial impulsion is small with respect to the initial deviation, the system, because of the presence of a large amount of friction, cannot pass the position of equilibrium and will approach it asymptotically (curve II). If the initial velocity is sufficiently high, the system at a given moment t_1 will cross the position of equilibrium (curve III) and will still retain a certain velocity directed away from equilibrium, i.e. in the same direction in which the system is moving. After that, the motion becomes of the type I already discussed; the system reaches first a certain maximum and then tends asymptotically to the position of equilibrium. Thus the motions of type III differ from those of type I only in their first part (previous to the time t_1), after which they are like motions of type I. On the other hand the motion of type I is like that of type II after the time t_2. In fact, the motion of the representative point along certain paths passing through the three regions I, II, and III (for example in Fig. 20) will belong either to type III or to type I, or to type II, according to the initial location of the point.

We shall not discuss the limiting case $h^2 = \omega_0^2$ in detail, for, like any other for which the relationship between the parameters of the system is exact (too strict), it cannot be reproduced exactly in a physical system and is of importance only as a border line between two different types of damping processes, the oscillatory and the aperiodic. The solution of the initial differential equation (13) will then be of the form

$$x = (A + Bt)e^{-qt}.$$

It is possible here also to dispense with the solution of the differential equation of the second order and to pass directly to the equation of the first order which determines the phase curves (25). We obtain a family of paths of parabolic type and a stable singular node. With regard to the behavior of the paths and the singular point, this limiting case should be correlated to the case $h^2 > \omega_0^2$ and not to the case $h^2 < \omega_0^2$. While the case $h^2 = \omega_0^2$ has no physical significance, it has, however, a certain mathematical interest. For it is often advantageous to choose the damping of the system so that h^2 will be as close as possible to ω_0^2. We thus eliminate the oscillations, inevitable when h^2 is much smaller than ω_0^2, making at the same time the velocity of aperiodic return of the system to zero maximum (this velocity is greater than for large values of h). These are the most advantageous conditions for a number of measuring devices, for example, for galvanometers. With the smallest variation in the parameters of the system, this limiting case will change into one of the two others.

It is important to keep in mind that the division of systems into oscillatory and aperiodic, which for a linear system can be mathematically quite precise, does not have great physical significance because for large values of h the system loses its most important "oscillatory" characteristics, indeed even before h^2 reaches ω_0^2. In fact, if h^2 is slightly smaller than ω_0^2, the damping is very important; already the second maximum, following the initial deviation, can be practically unnoticeable. Resonance, the most characteristic phenomenon in oscillatory systems, becomes also unnoticeable. Thus, although formally $h^2 = \omega_0^2$ is the limiting case, the actual limit between oscillatory and aperiodic processes is fuzzy and cannot be sharply defined. As we shall see later for certain non-linear systems (systems with constant friction, for example), the division into oscillatory and aperiodic systems becomes meaningless.

§5. Degenerate Linear Systems

1. Complete and "reduced" equations. In the equation describing a linear system with friction or resistance

$$m\ddot{x} + b\dot{x} + kx = 0, \tag{11}$$

there enter three coefficients or parameters. Each of these may be so small that the corresponding term is near zero, if the corresponding variable (x, $\dot{x}$, or $\ddot{x}$) does not reach too great a value. We may investigate these motions, neglecting the term whose value is small, i.e. describe them by an equation containing only two of the three in

(11). While such an idealization simplifies the mathematical problem, it eliminates the possibility of treating certain questions where the role of the neglected term is important. Therefore to decide whether any particular term may be disregarded one must compare the solutions obtained, taking into account the three parameters with the solutions obtained when the parameter in question is omitted. If the term $h\dot{x}$ is small, we have an equation of second order previously examined, describing a harmonic oscillator. If one of the two other terms $m\ddot{x}$ or kx is small (the mass or the spring constant is small), then, neglecting this term, we have

$$b\dot{x} + kx = 0 \tag{31}$$

or

$$m\ddot{x} + b\dot{x} = 0.$$

Let us examine first what happens if we assume $m = 0$ in (11) thus replacing (11) by (31). The solution of (31) has the form

$$x = Ae^{-\frac{k}{b}t},$$

or, introducing the initial conditions $t = 0$, $x = x_0$, the form

$$x = x_0 e^{-\frac{k}{b}t}, \qquad \dot{x} = -\frac{k}{b} x_0 e^{-\frac{k}{b}t}. \tag{32}$$

Naturally, the solution contains only one arbitrary constant since we are dealing with a differential equation of the first order. In this case, to determine uniquely the state of the system one needs to know only one element instead of the two required by an equation of the second order. When we neglect one of the oscillatory parameters of an ordinary system, we arrive at systems with "a half degree of freedom" sometimes called *degenerate* systems. The phase space relative to degenerate systems is one-dimensional—it represents a line and not a plane, and the position of the representative point on the "phase line" is defined uniquely by one coordinate.

Let us compare the solution (32) with the one obtained in §4 for the "complete" (i.e. non-degenerate) system, assuming that m is small but different from zero. If the initial conditions are $t = 0$, $x = x_0$, $\dot{x} = \dot{x}_0$, we obtain according to (27) the solution in the form:

$$x(t) = x_0\left(\frac{q_2}{q_2 - q_1} e^{-q_1 t} - \frac{q_1}{q_2 - q_1} e^{-q_2 t}\right) + \frac{\dot{x}_0}{q_2 - q_1}(e^{-q_1 t} - e^{-q_2 t}), \tag{33}$$

$$q_1 = \frac{b}{2m} - \sqrt{\frac{b^2}{4m^2} - \frac{k}{m}}, \qquad q_2 = \frac{b}{2m} + \sqrt{\frac{b^2}{4m^2} - \frac{k}{m}}.$$

To facilitate the comparison we replace the exact solution $x(t)$ of (11) by an approximate solution $x_1(t)$ such that, by choosing m sufficiently small, both $x_1 - x$ and $\dot{x}_1 - \dot{x}$ can be made as small as we choose, uniformly with respect to t.

Using the development of the radical

$$\sqrt{\frac{b^2}{4m^2} - \frac{k}{m}} = \frac{b}{2m}\sqrt{1 - \frac{4km}{b^2}} = \frac{b}{2m}\left(1 - \frac{2km}{b^2} + \cdots\right),$$

of which we only keep the first two terms, we have

$$(34) \qquad x_1(t) = x_0\left(e^{-\frac{k}{b}t} - \frac{mk}{b^2}e^{-\frac{b}{m}t}\right) + \dot{x}_0\frac{m}{b}\left(e^{-\frac{k}{b}t} - e^{-\frac{b}{m}t}\right).$$

One may show that the approximate solution is near the true solution in the sense that, whatever $\epsilon > 0$, we can always find m so small that

$$(35) \qquad |x_1(t) - x(t)| < \epsilon, \qquad |\dot{x}_1(t) - \dot{x}(t)| < \epsilon$$

for all positive t.[1]

Let us compare (32) and (34). Designating the solution of the degenerate equation by $\bar{x}$ and assuming that the initial values of the coordinates for the solution of the complete and of the degenerate equation are the same, we have

$$x_1(t) - \bar{x}(t) = -x_0\frac{mk}{b^2}e^{-\frac{b}{m}t} + \dot{x}_0\frac{m}{b}\left(e^{-\frac{k}{b}t} - e^{-\frac{b}{m}t}\right).$$

We can immediately see that this difference can be made as small as we choose by taking m sufficiently small, and this uniformly with respect to t for all positive t. On the other hand, the situation will be different for the derivatives. In fact,

$$(36) \qquad \dot{x}_1(t) - \dot{\bar{x}}(t) = +x_0\frac{k}{b}e^{-\frac{b}{m}t} - \dot{x}_0\frac{mk}{b^2}e^{-\frac{k}{b}t} + \dot{x}_0e^{-\frac{b}{m}t}.$$

For small t and m this difference will be near $\dot{x}_0 + (k/b)x_0$, a value which does not decrease with m and so cannot be made small by suitably choosing m. Owing to the rapid decrease of $e^{-\frac{b}{m}t}$ for a fixed

[1] Notice that, if m is small enough, the inequalities (35) cannot be replaced by inequalities

$$(\alpha) \qquad \left|\frac{x_1(t)}{x(t)} - 1\right| < \epsilon, \qquad \left|\frac{\dot{x}_1(t)}{\dot{x}(t)} - 1\right| < \epsilon,$$

valid for all t. However, corresponding to any t on a given interval $\tau \leqslant t \leqslant T$, one may replace (35) by (α) by choosing a suitably small m.

$t \neq 0$ and decreasing m, we learn from (36) that, given $\epsilon > 0$ and a sufficiently small m, there is a $\tau > 0$ such that

$$|\dot{x}_1(t) - \dot{\bar{x}}(t)| < \epsilon$$

for all $t \geqslant \tau$.

Thus for very small m and all positive t, the coordinate $x(t)$ given by the true solution of the complete differential equation with arbitrary initial conditions ($t = 0$, $x = x_0$, $\dot{x} = \dot{x}_0$) differs as little as we please from the corresponding values of the coordinate $\bar{x}(t)$ given by the solution of the degenerate differential equation with the same initial value. As to the velocities, they differ as little as we please for all values of t above a certain τ which depends on m.[1]

Let us explain this result. The difference between the initial conditions in the solutions of a complete and a degenerate system represents the difference between the initial velocity of a non-degenerate system (which can be chosen arbitrarily) and that corresponding to a given initial deviation in a degenerate system.

If the sum $\dot{x}_0 + \frac{k}{b} x_0$, as well as m, is sufficiently small, the difference between the velocities in the complete and the degenerate equation will remain small for all values of t. If this difference is not small, we obtain the following picture. When m is sufficiently small, the velocity in the complete equation of motion changes very rapidly; after a short time τ it almost coincides with the velocity given by the solution of the degenerate equation. This rapid change in velocity will be studied in detail later and will lead to a formulation of the so-called "condition of jump." The variation of the coordinate during this time τ (determined by the complete equation, for example) obviously tends to zero with τ.

Let us examine another case of degeneration, when k instead of m becomes zero.

The solution of the degenerate equation with normal initial conditions has in this case the following form:

$$\bar{x}(t) = x_0 + \frac{m}{b}\dot{x}_0\left(1 - e^{-\frac{b}{m}t}\right).$$

[1] One may prove the following property: Given any $\epsilon > 0$, there exist τ, T ($\tau < T$) such that for all t in $\tau \leqslant t \leqslant T$ we have

$$\left|\frac{\bar{x}(t)}{x(t)} - 1\right| < \epsilon, \qquad \left|\frac{d\bar{x}/dt}{dx/dt} - 1\right| < \epsilon.$$

Calculating as before an approximate solution of the complete equation (with the same initial conditions) , we find:

$$x_1(t) = x_0 e^{-\frac{k}{b}t} + \frac{m}{b}\,\dot{x}_0(1 - e^{-\frac{b}{m}t}).$$

It is easy to see that, for sufficiently small k and all positive t, we have, as previously,

$$|x_1(t) - x(t)| < \epsilon, \qquad |\dot{x}_1(t) - \dot{x}(t)| < \epsilon$$

where $x(t)$ represents again the solution of the complete equation. Let us compare $x_1(t)$ with the solution $\bar{x}(t)$ of the degenerate equation. In this case the inequality

$$|\dot{x}_1(t) - \dot{\bar{x}}(t)| < \epsilon$$

will be satisfied for all positive t only if k is sufficiently small. The choice of a sufficiently small k does not affect the difference

$$x_1(t) - \bar{x}(t) = -x_0(1 - e^{-\frac{k}{b}t})$$

for all t, since this difference tends to x_0 when t becomes sufficiently large.

We may easily see, however, that for a given ϵ and T we can always choose such a k that the inequality

$$|x_1(t) - \bar{x}(t)| < \epsilon$$

will be satisfied for all values of t in the interval $0 \leqslant t \leqslant T$.

We leave it to the reader to draw conclusions regarding the relationship between the true solution of the complete equation with small k and the solution of the degenerate equation.

2. Relation to the initial conditions. Let us return to the case of small m. The solution of the complete system contains two arbitrary constants, while the solution of the degenerate system contains only one. Therefore we cannot choose arbitrarily the two initial values x_0 and $\dot{x}_0$. *Hence we have a conflict between the possible states of an idealized system and the given initial conditions of the system.* To avoid it, the system with two given initial values must be represented by an equation of order two and not of order one. In many cases, however, the number of initial conditions which can be given arbitrarily in a real physical system does not depend upon the magnitude of a parameter of the system. Therefore, if we are investigating a system in which two initial values x_0 and $\dot{x}_0$ can be given arbitrarily, then, even

if one of the oscillating parameters is small, we should still be unable to regard the system as degenerate, i.e. to define it by one differential equation of order one, because this is incompatible with the initial conditions.

We have already seen that, if we neglect the motion during its initial period, we may describe the system by one equation of the first order. The following question arises then: How can we suppress the conflict and reconcile this equation with the initial conditions, which are, generally speaking, incompatible with it? In other words, how does the system pass from the given initial conditions to the state compatible with the description by means of an equation of the first order? It is quite clear why the question arises. When we disregard the motion during the initial period, we must replace it by a certain assumption concerning the state of the system at the end of the period. We have already studied this problem analytically, so let us examine it from the physical standpoint.

As an example, take a body of small mass, moving in a strongly resisting medium under the effect of a spring. The equation of motion will be

$$m\ddot{x} + b\dot{x} + kx = 0 \tag{11}$$

where m is a small parameter. We can choose arbitrarily the initial coordinate x_0 and velocity $\dot{x}_0$. On the other hand, we have already seen that, after a certain time (the smaller m, the smaller the time), the motion in the system can be described by the equation of first order:

$$b\dot{x} + kx = 0. \tag{31}$$

Here, however, $\dot{x}$ cannot be chosen arbitrarily but is determined by the coordinate x. Now at the initial time, the system may well be in a state incompatible with (31), that is to say the initial and perfectly arbitrary values of $x, \dot{x}$ need not satisfy (31). After that, if m is sufficiently small, the divergence from the solution of (31) is rapidly decreasing and the system passes into a state which is approximately compatible with (31). Let us have, for example, at the initial moment $x = x_0$ and $\dot{x} = 0$, conditions clearly incompatible with the first order equation (31). As long as $\dot{x}$ is very small, the term $b\dot{x}$ will not be important and, according to (11), the acceleration will be given by the approximate expression

$$\ddot{x} \approx -\frac{k}{m}x.$$

Since m is very small, the acceleration is very large; the velocity increases very rapidly. At the same time, friction increases and a greater and greater part of the force of the spring is spent in overcoming friction. Consequently the acceleration becomes smaller and smaller, and finally the term $m\ddot{x}$ ceases to play any role. The motion of the system can then be satisfactorily described by the equation of the first order (31). At this point, the velocity $\dot{x}$ acquires a value depending on x, for when $m\ddot{x}$ disappears an approximate equality between kx and $(-b\dot{x})$ takes place. This is the way in which the rapid passage is accomplished from a state incompatible with (31) to a state compatible with it. We have already studied this passage analytically, using the complete equation of second order (11) and its solution (34).

Thus if m is sufficiently small, the acceleration is at first very large and the velocity varies very rapidly. After a very short time the system passes into a state compatible with the equation of the first order, and this time interval is so small that, in spite of a large acceleration, x does not have time to undergo an appreciable change.

3. Conditions for a discontinuity (jump). We have seen that. when the system passes into a state compatible with the equation of the first order, its velocity varies very rapidly while the coordinate remains almost constant. If this passage is sufficiently rapid, its details are often without interest. We may regard this rapid passage as an instantaneous jump and determine only the final state into which the system jumps; afterwards the behavior of the system is determined by the equation of the first order (31). We may therefore consider the system as mass-free provided that we introduce a new assumption, namely, that *there occurs a discontinuity.* In our particular case it could be formulated as follows: The velocity $\dot{x}$ changes abruptly while the coordinate x remains constant. Similarly, in a system with small k, we infer from our earlier discussion that the acceleration varies at first very rapidly while the velocity does not have time to change. Here we must assume that the acceleration undergoes a jump while the velocity remains constant. In a similar way, for degenerate electrical systems we should formulate special conditions for a jump. A general formulation, manifestly desirable, is as follows: *The energy of the system cannot undergo a jump, and hence the system can jump from one state to another only if it possesses the same energy in the initial and in the final state.* The meaning of this assumption can be brought out in the following way: The smaller the interval of time in which the energy increases by a given amount, the greater the corresponding power developed by the system; therefore, a jump-like variation of

energy is possible only in the presence of unlimited power. Our assumption means, then, that the physical systems under investigation are not supposed to develop unlimited power. It should be noted, however, that in studying the shock phenomena in mechanics it is sometimes necessary to introduce the notion of non-conservative jumps (for example, in the case of an inelastic shock) in which there occurs an instantaneous variation of energy. As we shall see later, analogous phenomena can take place also in electrical systems (see the theory of

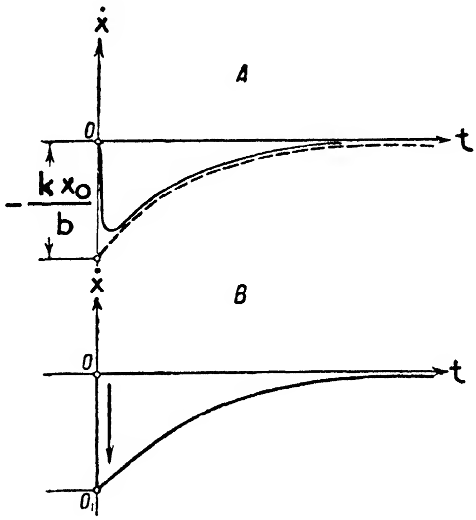

Fig. 22.

generators with step characteristic and oscillating circuits in a network, p. 331). In all these cases, the corresponding condition of the jump must include a quantitative characterization of this instantaneous change of energy. However, in dealing with processes which do not have this character, the assumption that energy cannot change instantaneously is entirely reasonable.[1]

We shall illustrate our assumption by physical examples. In the massless spring there is no kinetic energy since $m = 0$, and so the total energy is the potential energy $kx^2/2$ of the spring. Under the energetic assumption this energy must remain constant, and so x is continuous

[1] Assumptions such as those made here are often utilized when one wishes to avoid the detailed examination of a phenomenon. Thus in elastic shocks one assumes that the velocity of a body with mass changes instantaneously, which implies that the body is subjected to infinite forces. These are the elastic forces which are, in fact, finite so that the process is not actually instantaneous. However, since the change of velocity is very rapid and the intervening process of no interest, it is most convenient to introduce assumptions which make it possible to determine the velocity after the shock in terms of its value before.

but the velocity $\dot{x}$ may vary abruptly. It will then pass from its initial value to the value imposed by the equation of the first order (31) for a given initial value x_0 of x. Beyond the jump, motion proceeds continuously in accordance with (31).

Let us represent graphically the meaning of the condition of the jump. Since, in our case, the velocity undergoes a jump, we will compare the velocity diagram as a function of time for the case $m \neq 0$ (equation of second order) with a similar diagram for $m = 0$ (equation of first order plus condition of the jump). The initial values of x and $\dot{x}$ can be taken arbitrarily, say, for $t = 0$, $x = x_0$, $(x_0 > 0)$, $\dot{x} = 0$. The dependence of the velocity on time, following the equation of

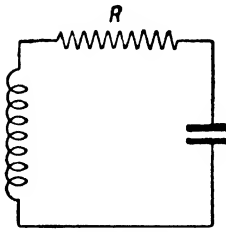

FIG. 23.

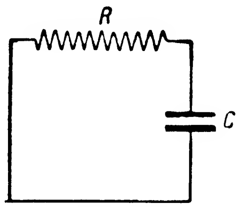

FIG. 24.

second order, has the form represented in Fig. 22A (taking $m \ll b^2/k$ in the construction of the graph).

If we use the equation of the first order, the initial value $x = x_0$ automatically provides the initial value $\dot{x} = -\frac{k}{b}x_0$ and a further variation of the velocity with time according to Fig. 22B.

The jump eliminating the conflict between the initial conditions $x = x_0$, $\dot{x} = 0$ and the differential equation of first order is represented in Fig. 22B by the cord OO_1. The similarity of Figs. 22A and 22B is evident; the physical meaning of this similarity has been explained earlier.

An electrical circuit including capacitance, inductance and resistance (Fig. 23) will serve as a second example of a degenerate system. In the equation describing the behavior of the circuit:

$$L\ddot{q} + R\dot{q} + \frac{q}{C} = 0, \tag{37}$$

let the parameter L be small while R is large. The values q_0 and $\dot{q}_0$ (charge on the condenser and current in the circuit) corresponding to the initial time can be chosen arbitrarily. Neglecting inductance, i.e. assuming $L = 0$, we obtain a circuit represented in Fig. 24, which is

described by the equation

$$R\dot{q} + \frac{q}{C} = 0. \tag{38}$$

Here $\dot{q}_0$ is defined by the initial value q_0. At the initial time, however, we may arbitrarily choose q as well as $\dot{q}$. Let us take, for example, $\dot{q}_0 = 0$. According to (38), $\dot{q}_0 = -\frac{q_0}{RC}$ when $q = q_0$ ($q_0 > 0$). The initial current can be chosen arbitrarily. It does not correspond to the value given by this equation. However, since the inductance is very small, the variation of the current is very rapid and the current approaches very rapidly the value which would be given by the equation of the first order (38). The charge on the plates of the condenser, however, does not have time to change during this period. Using the complete equation (37), we may follow these rapid variations of the current and obtain a curve $\dot{q} = f(t)$ analogous to $\dot{x} = f(t)$ represented in Fig. 22A. If we are not interested in the details of the variation of current for small t, we may neglect inductance and, instead of studying the initial stages of the movement, introduce the jump. As long as we consider that the circuit possesses only capacitance and does not have inductance, we must assume that all the energy of the circuit is stored in the condenser. Hence the condition of jump permits, in this case, an abrupt variation of the current with the charge of the condenser remaining constant. When we neglect inductance we must assume that, independently of the initial current i_0, the current i jumps to a value defined by the equation of the first order (38). We then have for $i = \dot{q}$ the same curve as the one for $\dot{x}$ in Fig. 22B. Of course, a real circuit always has some inductance, thus ruling out abrupt jumps of the current. If, however, the inductance is small and the current changes rapidly, we may assume for many applications that it undergoes an instantaneous jump.

If we assume instantaneous variation of the current in a circuit containing inductance, i.e. if we assume that at certain times $\ddot{q} = \infty$, we must also assume an infinite induced voltage $L\ddot{q}$ across the coil. On the other hand, if we assume instantaneous variation of the charge on the condenser, we are forced to admit the creation of infinitely large currents in the circuit (because, if q changes by a jump, $\dot{q} = i = \infty$). Our assumption concerning the character of the jump does not allow either of these variations.

An analogous discussion can be applied to a circuit with inductance and resistance but without capacitance (Fig. 25). The concept of

such a circuit is also an idealization. The equation of the circuit is now

$$L\dot{i} + Ri = 0. \tag{39}$$

The initial values $i = i_0$ and $\dot{i} = \dot{i}_0$ can be taken arbitrarily. According to (39) the relationship between i and $\dot{i}$ is unique. Consequently, we must assume a sudden variation of $\dot{i}$ at the initial time with the observation of the condition of the jump according to which i must remain constant. All the energy of the circuit is stored in the coil and is equal to $Li^2/2$. After that, the motion is determined by (39) and proceeds continuously (without jump).

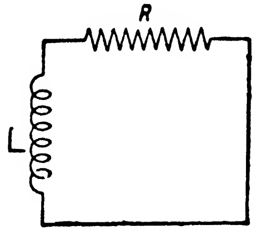

FIG. 25.

4. Stray parameters. We have reduced the order of the basic differential equation from two to one by neglecting inductance. It could also be done by neglecting capacitance, and this would be as important from the practical standpoint. It is, however, easier to construct a circuit in which inductance plays a secondary role than one in which the secondary role is played by capacitance. Theoretically, however, both have equal weight. Both may lead to the notion of jump and, conversely, the presence of small inductance or capacitance can prohibit these jumps. These small inductances and capacitances are present in all real systems and cannot be totally eliminated. Whenever they are not consciously introduced into the scheme but are present as the result of properties of the conductors, they are called stray capacitance and inductance and they are usually neglected. Thus, in order to obtain the circuit with one degree of freedom of Fig. 25, we have to neglect the stray capacitance of the induction coils and of the resistances. If we take into account the stray capacitance, even under the form of a lumped capacitance C_0 (Fig. 26), we see immediately that $\dot{i}$ cannot vary by a jump (the voltage across the stray capacitance is $L\dot{i}$; it cannot undergo a jump because then the energy of the charge of the condenser C_0 would also have to undergo a jump). Similarly, if we take into account the stray capacitance (as a lumped capacitance C_0) of the resistance of the circuit represented in Fig. 24, we shall arrive at the circuit represented in Fig. 27. We can immediately see that in this scheme the jumps of the current become impossible because the voltage across C_0 is equal to Ri and cannot vary by jumps.

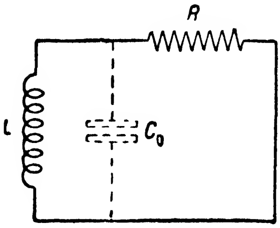

FIG. 26.

Let us note a circumstance which, at first, does not seem to be clear. In Fig. 27 the condensers C and C_0 are in parallel, which may be regarded as a simple increase of the capacity C. Why, then, does the capacity C_0 exclude jumps, while the capacity C does not? This *apparent difficulty disappears* immediately if we realize that C_0 is a stray capacitance due to the mutual capacitance of the turns of the resistance; therefore we can neither eliminate it nor divorce it from the resistance R. Hence we cannot choose any of the initial conditions of the "circuit" C_0,R. The only initial characteristics of this arrangement that we can choose are the initial current i_0 and the initial charge q_0 of the plates of the condenser C_0. We are able to give independent

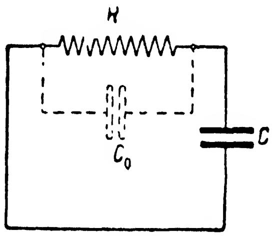

FIG. 27.

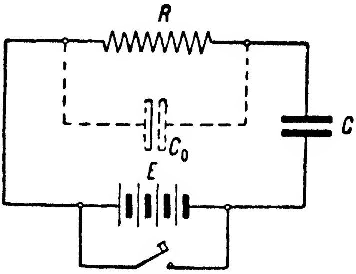

FIG. 28.

values to i_0 and q_0 if, for example, we introduce between C and R a certain difference of potential E (Fig. 28). Even under these simple circumstances we do introduce initial conditions incompatible with the equation of first order

$$Ri + \frac{q}{C} = 0$$

when the switch in Fig. 28 is closed.

As we cannot disconnect the stray capacitance, in the circuit of Fig. 27, described by two differential equations of first order with two integration constants, we can introduce arbitrarily only two initial values. Thus in this case the possibility of a conflict caused by incompatibility between the initial conditions and the equation, and related sudden variations in the state of the system, is eliminated. By taking into account the stray capacitance, not only are we introducing a certain additional capacitance C_0 but also a certain additional condition (connecting q_0 and q_{00}, the initial charge on C_0) which makes the jumps impossible and eliminates a possible conflict.

At this juncture it is natural to ask the following question: What modifications in the stability behavior are associated with the replace-

ment of a non-degenerate differential equation of order two by a degenerate equation of order one? One can see at once that, if in the solution (33) one of the exponents increases indefinitely (as a consequence of the degeneracy), the states of equilibrium are stable in both the degenerate and the non-degenerate system, and, as both exponents are negative, stability is unaffected. Later we shall discuss cases where stability is affected.

§6. Linear Systems with "Negative Friction"

In ordinary systems with friction, the coefficient $h = b/2m$ (or in electrical systems $h = R/2L$) is always positive since friction always resists the motion and $b > 0$ ($R > 0$). A positive coefficient of friction and a positive resistance mean that the overcoming of friction (or resistance in electrical circuits) requires energy. In fact, if in the equation of motion

$$m\ddot{x} + b\dot{x} + kx = 0 \tag{11}$$

we multiply all the terms by $\dot{x}$ and integrate from 0 to τ, we obtain

$$m\int_0^\tau \ddot{x}\dot{x}\,dt + \int_0^\tau b\dot{x}^2\,dt + \int_0^\tau kx\dot{x}\,dt = 0.$$

By integration we find

$$\left|\frac{m\dot{x}^2}{2}\right|_0^\tau + \left|\frac{kx^2}{2}\right|_0^\tau = -\textstyle\int_0^\tau b\dot{x}^2\,dt.$$

At the left we have terms representing the variation of the kinetic and potential energies of the system when t goes from 0 to τ; their sum is the variation of the total energy during this time. If $b > 0$, the integral at the right is positive and the variation of energy is negative, i.e. the energy of the system decreases. This decrease of energy is caused by the loss of energy due to friction.

If b and hence h were negative, the energy of the system would increase and "friction" would represent a source of energy. It is clear that in a system without a source of energy this is impossible, and b as well as h is always positive. If the system possesses a reservoir of energy, then one can allow $h < 0$ and the energy of the system increases on account of "friction" or "resistance." Of course this will no longer be friction or resistance in the ordinary sense. When it is characterized by the same term of the differential equation as ordinary friction, namely by the term in $\dot{x}$, we shall use the expression "friction and resistance" even in the case of negative h, and speak of "negative friction" and "negative resistance."

1. Mechanical example. The simplest example of a mechanical system where "friction" is negative in a certain region is represented in Fig. 29. A mass m fixed by the springs k_1 and k_2 is placed on a ribbon moving uniformly with a velocity v_0. The friction of the ribbon on the mass is a certain, quite complicated function $F(u)$ of their relative velocity u. Let x denote the displacement of the mass and $\dot{x}$ its velocity; then the friction on the mass m is expressed by $F(v_0 - \dot{x})$. If we designate the "resulting" spring constant by k and assume that all the other frictional forces (for example, air resistance or internal friction of the springs) are proportional to the first power of the velocity, the equation of motion of the mass m is of the form

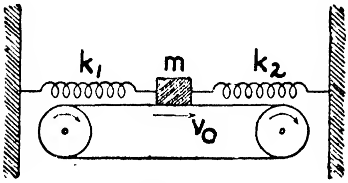

FIG. 29.

$$m\ddot{x} + b\dot{x} + kx = F(v_0 - \dot{x}). \tag{40}$$

We may confine our attention to the region where $\dot{x} \ll v_0$. Assuming $\dot{x}$ small, let $F(v_0 - \dot{x})$ be expanded in powers of $\dot{x}$

$$F(v_0 - \dot{x}) = F(v_0) - \dot{x}F'(v_0) + \cdots .$$

Keeping only the first two terms and substituting in (40), it assumes the form

$$m\ddot{x} + [b + F'(v_0)]\dot{x} + kx = F(v_0).$$

The position $x_0 = \dfrac{1}{k}F(v_0)$ corresponds to $\dot{x} = \ddot{x} = 0$, i.e. it is a position of equilibrium. Replacing x by $x_0 + x$ amounts to measuring the displacement from the position of equilibrium and puts the equation in the form

$$m\ddot{x} + [b + F'(v_0)]\dot{x} + kx = 0. \tag{41}$$

As to the coefficient $b + F'(v_0)$ of $\dot{x}$, its value and sign depend on the form of the characteristic of friction; $F'(v_0)$ represents the slope of the frictional characteristic $F(u)$ at the point v_0. If the frictional characteristic decreases, $F'(v_0) < 0$. If the frictional characteristic decreases very rapidly about v_0, then $b + F'(v_0) < 0$ and (41) describes a system with "negative friction." It is easy to realize this case in practice because frictional characteristics of dry surfaces have generally the form represented in Fig. 30; there is almost always at the beginning a part where they decrease sufficiently rapidly for small velocities. In this part our scheme will represent a linear

system with "negative friction." One has to note that we arrived at such a linear system by limiting ourselves to $\dot{x} \ll v_0$. This limitation will play an important role in a question of interest later.

Another example of a mechanical system in which friction is negative in a certain part is the Froude pendulum. This is an ordinary pendulum suspended with friction on a shaft moving uniformly with an angular velocity Ω (Fig. 31). The equation of motion of this pendulum only differs from that of an ordinary pendulum in that the moment of the frictional force of the revolving shaft against the bearing has to be accounted for. Since the frictional force depends upon the relative velocities of the rubbing surfaces, i.e. in our case upon the relative angular velocity $(\Omega - \dot{\phi})$ of the shaft and of the pendulum, the moment of the frictional force can be expressed by $F(\Omega - \dot{\phi})$. Let us limit, as usual, the discussion to small angles ϕ, replacing $\sin \phi$ by ϕ, and let us take into consideration, besides the friction of the pendulum against the shaft, air resistance assumed proportional to the angular velocity. Then the equation of motion will be

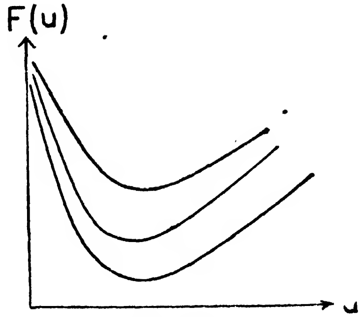

FIG. 30.

$$I\ddot{\phi} + b\dot{\phi} + mgl\phi = F(\Omega - \dot{\phi}). \tag{42}$$

Confining our attention, as in the preceding example, to $\dot{\phi} \ll \Omega$, we develop $F(\Omega - \dot{\phi})$ in series of powers of $\dot{\phi}$ and retain only the first term. Hence

$$I\ddot{\phi} + b\dot{\phi} + mgl\phi = F(\Omega) - \dot{\phi}F'(\Omega). \tag{43}$$

The constant term $F(\Omega)$[1] causes here also merely a displacement of the equilibrium. For the new angular coordinate, still called ϕ, taken from the new displaced position of equilibrium, the equation of motion assumes the form

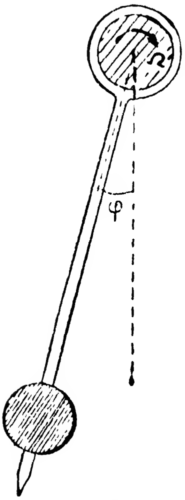

FIG. 31.

$$I\ddot{\phi} + [b + F'(\Omega)]\dot{\phi} + mgl\phi = 0. \tag{44}$$

[1] Notice that $F(\Omega)$ may be assumed constant merely in so far as the pressure of the bearing on the shaft during the oscillation of the pendulum can be assumed constant and in so far as one may neglect the redistribution of pressure, for example when the bearing is so tight that one may disregard the pressure due to the weight of the pendulum.

If $F'(\Omega) < 0$ and its absolute value is greater than that of b, the coefficient of $\dot{\phi}$ will be negative. In a certain interval of Ω values where the frictional characteristic decreases very rapidly for sufficiently small b, it is possible to have $b + F'(\Omega)$ negative and we obtain then an equation analogous to the equation of an ordinary system with friction

$$\ddot{\phi} + 2h\dot{\phi} + \omega_0^2\phi = 0$$

with the difference that the coefficient h will be negative, i.e. we shall again obtain a linear system with "negative friction."

2. Electrical example. It is entirely possible to construct an electrical system which will have a "negative resistance" in a certain interval. A vacuum tube oscillator, represented in Fig. 32, i.e. a scheme comprising a vacuum tube, an oscillatory circuit, and *feedback*, is such a system. Its distinguishing feature is a coupling between the grid and the plate, called *feedback coupling.* It can be obtained in various ways.

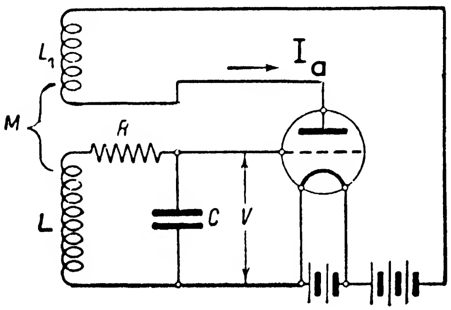

FIG. 32.

We shall study the simple case of a circuit consisting of an inductive feedback coupling;[1] we shall not examine circuits including capacitance or mixed feedback coupling, for either they do not offer anything essentially new or else they lead to differential equations of the third order, i.e. to a system with "one and a half degrees of freedom" and thus exceed the scope of this book. Let us assume for simplicity that there is no grid current in the circuit.

For the grid circuit (in the scheme with feedback coupling) we obtain an equation differing from the ordinary equation of oscillatory circuits only in that the right-hand part of the equation will include an input voltage. It arises in the circuit as a consequence of the action of the plate current through the coil L_1 on the circuit. Its expression is $M\dot{I}_a$, and the equation of the circuit is

$$LC\ddot{V} + RC\dot{V} + V = M\dot{I}_a. \tag{45}$$

If we neglect the resistance in the capacitance link of the circuit, we can assume that the difference of potential V on the condenser and

[1] A similar treatment could be applied to a circuit with inductive feedback coupling but with a tuned plate circuit.

on the grid is the same and therefore I_a is a function of V. Generally speaking, the plate current is a complicated function of the plate and grid voltages. For simplicity we shall neglect the effect of the plate voltage and assume that I_a is a function of V alone. Under these assumptions we may write $I_a = f(V)$ and hence

$$\dot{I}_a = g_m \dot{V} \tag{46}$$

where $g_m = df/dV$, the mutual conductance of the tube g_m is a variable depending on V in a complicated manner. An example of this characteristic is represented in Fig. 33.

Introducing the expression (46) into the expression (45) we obtain

$$LCV + (RC - Mg_m)\dot{V} + V = 0.$$

Thus we arrive at an equation in which, instead of R, we have a more complicated expression which is, in general, variable since g_m depends on V. If we choose a portion of the characteristic which can be considered linear, for example, the portion between a and b (Fig. 33), and examine the behavior of the system in a limited range of voltages, from V_a to V_b, we may assume that g_m is constant. In other words, we can choose such a limited range for V that, when developing $f(V)$ in series, we need only retain the linear part. Then $g_{m_0} = (\partial f/\partial V)_{V_0}$ = the mutual conductance at the operating point. M is determined by the relative position of the turns of the coils L_1 and L (see Fig. 32). Assume that it corresponds to a positive M, i.e. positive feedback. Then, if M is sufficiently large, $RC - Mg_{m_0}$ may be negative. Thus we obtain an electrical system described by a linear equation

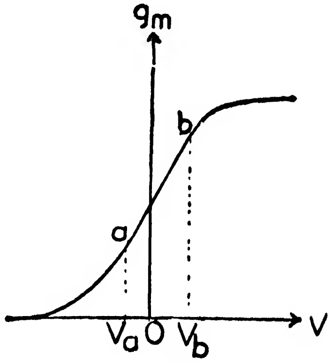

FIG. 33.

$$\dot{V} + 2h\dot{V} + \omega_0^2 V = 0$$

with $h < 0$, i.e. with "negative resistance."

3. Representation on the phase plane when the resistance is negative. We have just considered phenomena with "negative" resistance, i.e. leading to a differential equation

$$\ddot{x} + 2h\dot{x} + \omega_0^2 x = 0, \qquad h < 0.$$

Their treatment is formally the same as when $h > 0$. The practical differences occur only between h quite small: $h^2 < \omega_0^2$ and h quite large: $h^2 > \omega_0^2$. We discuss the corresponding paths separately.

First case: $h^2 < \omega_0^2$. The curves are spirals represented as before by (20) (cartesian coordinates) or (23) (polar coordinates), and the sense of description is illustrated in full by Figs. 34 and 35. The representative point moves away from the origin, and the state of equilibrium corresponding to the origin is unstable. The origin is an

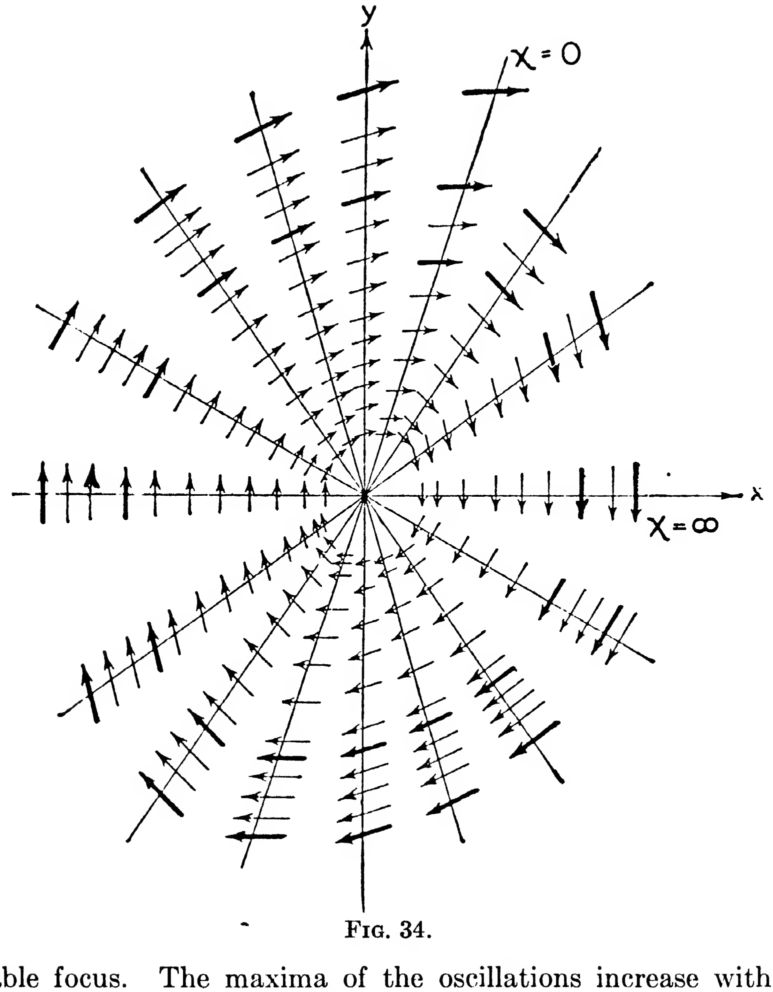

FIG. 34.

unstable focus. The maxima of the oscillations increase with time (Fig. 36) and form a geometric progression of ratio $e^{-hT_1} = e^d$, $d > 0$. The number d is the *logarithmic increment* of the system.

Second case: $h^2 > \omega_0^2$. The paths are represented by (30). This time, since $h < 0$, the exponents q_1,q_2 are such that $q_1 < q_2 < 0$. The curves are parabolic as before and shown in Fig. 37. The representative point moves again away from the origin, which is now an unstable node.

An important observation is to be made regarding the physical systems which gave rise to a "negative" resistance: the corresponding differential equations turned out to be linear solely under the assumption that the coordinate x and velocity $\dot{x}$ are not very large. In terms

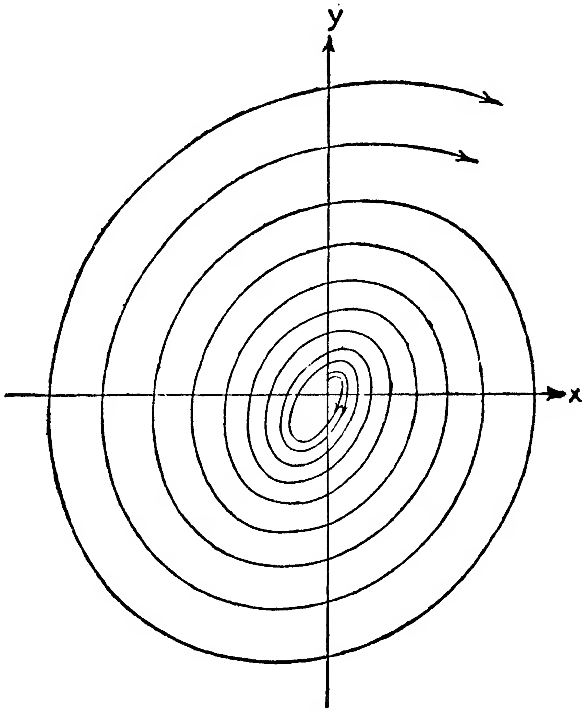

FIG. 35.

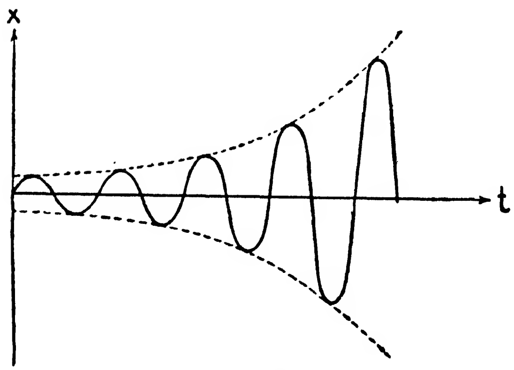

FIG. 36.

of the phase plane this assumption means that the representative point remains quite near the origin. From the practical standpoint the chief information that we have gathered is that the point tends to depart from the origin. When it reaches a position outside a certain region,

the assumption at the root of linearity ceases to hold and one must have recourse to some other differential equation to describe the motion. In the mechanical example leading to (42) this shows itself in that when ϕ is large one must take terms of higher degree than one in the development of $F(\Omega - \phi)$ and thus (42) will not yield (43) nor (44) but an equation with terms in ϕ^2 and also perhaps in ϕ^3 and higher powers of ϕ, in short a non-linear differential equation. Similarly

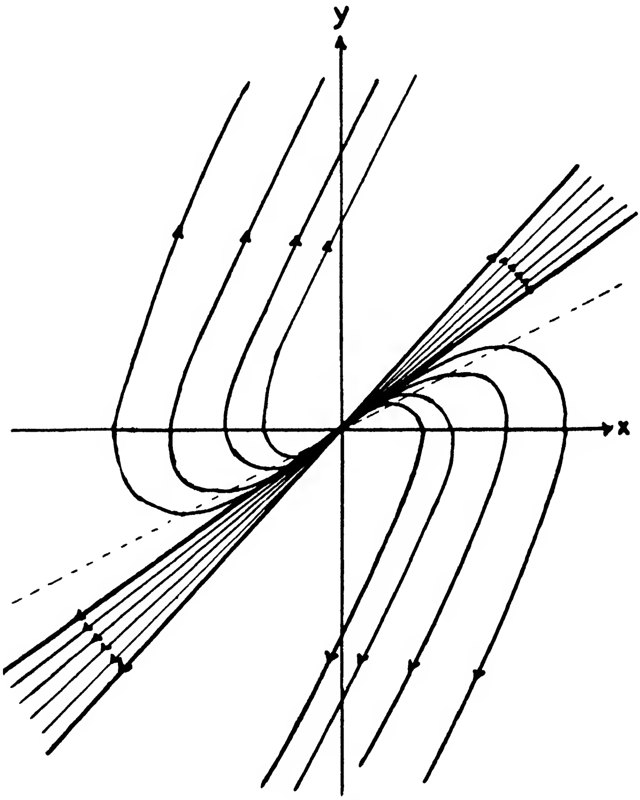

FIG. 37.

with the electrical example and (45). In other words, *in general a linear equation may not suffice to describe a phenomenon throughout an indefinitely large time interval.* The necessity of studying non-linear systems becomes thus apparent. They will be taken up at length in the following chapters.

4. Behavior of systems with changing feedback coupling. Referring everything for convenience to the electrical model with feedback coupling,[1] we continue to suppose the coil so arranged that $M > 0$.

[1] In mechanical systems there is no analogue to variable feedback coupling.

We have then $h = (RC - Mg_{m_0})/2L$ and, by compensating for the resistance by the coupling to as high or low an extent as we please, we may make h decrease from large positive values to ω_0, then to zero, then through negative values to $-\omega_0$ and below. We shall then pass successively through all the systems considered in succession: stable node, stable focus, center (for $h = 0$), unstable focus, unstable node. Each of the states except the center is characterized by a definite interval of values for h. The center corresponds to a single value $h = 0$ and is not realizable physically but rather to be viewed as a limiting case between the stable and unstable foci.

The passage from one type to another proceeds gradually in real systems, and the physical limit between aperiodic and oscillatory damping is not too sharp. For, when damping increases, the system loses its oscillatory property gradually and not all at once. In other words, in real systems we cannot distinguish a "strong" focus, i.e. focus with a very large h (when h^2 is only a little smaller than ω_0^2) from a "weak" node, i.e. a node for which h^2 is only slightly larger than ω_0^2. In the same way, we cannot distinguish very weak damping from very weak growth because in order to discover the difference between these two processes we would have to wait for an extremely long period of time.

As we have seen above, an adequate choice of the direction and magnitude of feedback enables us not only to decrease the damping of proper oscillations of the system but also to make the oscillations grow. The physical meaning of this phenomenon is entirely clear. The decrease in damping is apparently conditioned by the acquisition of energy from an outside source (in our case the plate supply source) and it compensates part of the energy dissipated in the circuit, thus slowing down damping. The stronger the feedback, the more energy enters from the battery into the circuit for a given period, and the greater the compensated part of the energy loss becomes, and consequently the weaker the damping of the oscillations. When the feedback increases further, the energy entering the circuit may become

---

Hence to modify the situation one must have recourse to some other parameter such as the friction coefficient. In a regenerative circuit, instead of modifying the magnitude of feedback coupling, one could change the transconductance characteristic at the working point, i.e. the value of g_{m_0}. The absence of feedback coupling in mechanical systems makes the analogy with vacuum tube generators incomplete. The analogue of a pendulum may be obtained by means of a dynatron without feedback coupling where self-excitation is caused by work on the downward portion of the characteristic of the vacuum tube.

greater than the energy lost, and then the energy in the circuit will increase—growing oscillations will take place in the circuit and, if the feedback increases further, even aperiodic growth can take place. As to the duration of the growth of oscillations, we cannot say anything as long as we limit ourselves to a linear scheme.

Let us return to the question of compensation of energy losses in the system. For electrical as well as for mechanical systems the picture from this point of view is the same. In the case of an oscillator, the energy is furnished to the circuit by a d.c. plate supply, and the tube represents only the mechanism regulating the input of energy into the circuit. In the mechanical system of No. 1, to which all our conclusions can be applied, the motor moving the ribbon or the shaft is the source of energy, while the transmission of this energy to the oscillatory system is conditioned by a corresponding form of the characteristic of friction. The form of the characteristic is such that the belt and the shaft "help" the body move in one direction more than they hamper it when it moves in the opposite direction. If we have $M < 0$ (feedback coupling) for the regenerating receiver, or if in mechanical systems we situate the operating point on the upward instead of the downward region of the friction characteristic, then the energy of the battery or the motor would not pass into the oscillatory system; on the contrary, part of the energy of oscillations would dissipate in an auxiliary mechanism (at the anode or at the bearing) to overcome friction. The damping of oscillations in the system would not only decrease but, on the contrary, would increase if the direction of the feedback coupling were incorrect.

To conclude, let us note (although these questions will not be studied in this book) that when an outside force acts upon a system with feedback (for example, upon a regenerative receiver) it is possible, retaining the linear idealization, to answer certain questions. For example, when $h < 0$ (i.e. in the case of an under-excited regenerator) and the signals are weak (i.e. the case of an action not driving the system out of the region which can be regarded as linear), the feedback only decreases the damping of the system (increases its sensitivity and selectivity) without changing the "linearity" of the properties of the system. For sufficiently strong signals, however, this assumption would be incorrect.

§7. LINEAR SYSTEM WITH REPULSIVE FORCE

Thus far we have been studying linear systems with a quasi-elastic force, i.e. a force "attracting" to the equilibrium position and propor-

tional to the displacement of the system. In all these the character of friction varied but the force remained attractive. Quite often, however, one meets with systems of considerable interest to theory of oscillations, in which the force, instead of attracting the system to the position of equilibrium, repels it away from it, the magnitude of this repulsive force increasing with the displacement. When studying such systems the question arises: What is the relationship between the repulsive force and the displacement? We shall examine a few examples and find that in the region of sufficiently small deviations one can consider the repulsive force as proportional to displacement. Such an assumption leads to linear systems having a repulsive force instead of an attractive force. The behavior of these systems differs substantially from the behavior of the linear systems investigated above.

FIG. 38.

As a first example, let us examine the behavior of a mathematical frictionless pendulum in the immediate vicinity of the upper (unstable) equilibrium position. Then, if the angle ϕ is measured from the upper position of equilibrium (Fig. 38), the equation of motion will be

$$ml^2\ddot{\phi} = mgl \sin \phi.$$

If we limit our observations to a small neighborhood of the position of equilibrium, we can replace $\sin \phi$ by ϕ. The equation will then take the linear form

$$\ddot{\phi} - \frac{g}{l}\phi = 0. \tag{47}$$

This equation, as well as the similar one for the lower position of equilibrium, does not describe the motion of the pendulum for all values of the angle ϕ, but only for sufficiently small values of ϕ.

1. Representation on the phase plane. The equation (47) just obtained is of the general type

$$m\ddot{x} - kx = 0 \tag{48}$$

where m and k are positive. To discuss the behavior of the system described by this equation, we may apply any one of our earlier methods. Thus we may integrate (48) and then discuss the solution $x = f(t)$ and $\dot{x} = f'(t)$ as parametric equations of the paths; or else, without integrating (48), we may eliminate the time and then integrate the resulting equation, considering it as the equation of the paths. Let

us use the second method. Taking $\dot{x} = y$, we can replace (48) by two equations of the first order

$$\frac{dx}{dt} = y; \qquad \frac{dy}{dt} = \frac{k}{m}x;$$

from which follows

$$\frac{dy}{dx} = \frac{k}{m}\frac{x}{y}. \tag{49}$$

The system has only one state of equilibrium (given, as usual, by $dx/dt = 0$ and $dy/dt = 0$), namely, the point $x = 0$, $y = 0$. The y-axis is the isocline $x = 0$ $(dy/dx = 0)$ and the x-axis the isocline $y = 0$ $(dy/dx = \infty)$. In order to determine the exact form of the paths, one must integrate (49). The variables can be separated, and the integration yields

$$\frac{y^2}{k} - \frac{x^2}{m} = C.$$

This is the equation of a family of equilateral hyperbolas symmetric with respect to the axes. If $C = 0$, we have the asymptotes of the family: $y = -\sqrt{k/m}\,x$ and $y = \sqrt{k/m}\,x$. The origin is the only singular point of the family of paths. A singular point of this type to which there tend only paths which are asymptotes (the others being hyperbolas which do not pass through the singular point) is called a *saddle point*.

Bearing in mind that x increases when the velocity $\dot{x}$ is positive and decreases when $\dot{x}$ is negative, we obtain in the four quadrants the directions of motion of the representative point over the phase plane, as shown by the arrows in Fig. 39. Clearly, whatever the initial position of the representative point (with the exception of the singular point and of the points on the asymptote $y = -\sqrt{k/m}\,x$ passing through the second and fourth quadrants), it always moves away from the state of equilibrium and the motion will be aperiodic and not oscillating. The velocity of the representative point becomes zero only at the singular point, and it is different from zero elsewhere. Thus, even if at first the point moves along one of the paths towards the singular point (motion in the second and fourth quadrants), it will move later as far as we choose away from it. This will take place in all cases with the exception of motion along the asymptote $y = -\sqrt{k/m}\,x$. Hence here equilibrium is unstable, for, clearly given a region ϵ, there is no corresponding region $\delta(\epsilon)$ such that, if the represent-

ative point is ever in $\delta(\epsilon)$, it does not leave the region ϵ. Thus a saddle point is always unstable. This instability is related to the type of singular point and the character of the paths and not to the direction of motion of the representative point along the paths (if the motion is reversed, the singular point will remain unstable).

As to the motion along the asymptote $y = -\sqrt{k/m}\,x$, it represents a special case when the system can only tend to the equilibrium

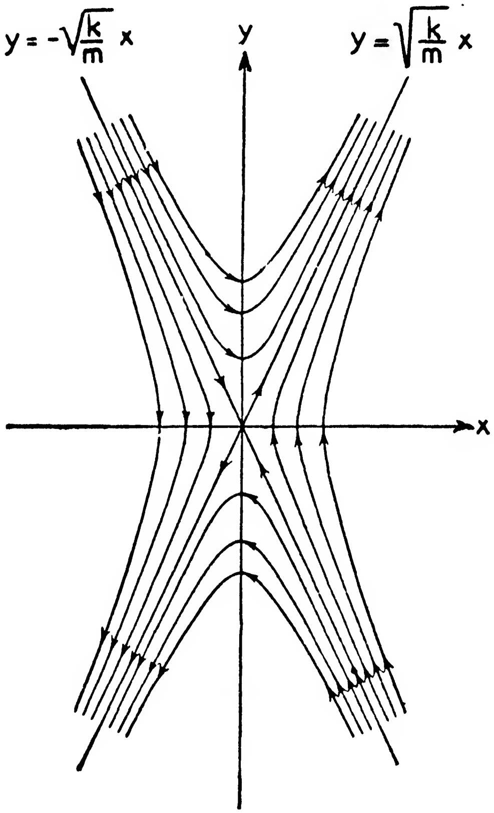

FIG. 39.

state. During this motion, the representative point tends to the origin with a velocity tending to zero but never reaches it. This type of motion will be studied later in detail. The possibility of such a motion toward the state of unstable equilibrium is, however, quite clear on elementary grounds. In fact, for all initial deviations of the pendulum from the upper state of equilibrium, we may always choose the initial velocity so that the kinetic energy of the pendulum at the initial moment is exactly equal to the work it has to spend in order to reach exactly the upper state of equilibrium. As we shall see later, however,

if we could choose exactly such an initial velocity, the pendulum would reach the upper state of equilibrium only after an unlimited time.

Let us suppose now that, in addition to the repulsive force, the system possesses friction, which is either positive or negative. The first case will take place when the pendulum is near the upper equilibrium position and the friction force is proportional to the velocity. In this case, the equation describing the system will have the form

$$\ddot{\phi} + 2h\dot{\phi} - k\phi = 0 \tag{50}$$

where $h > 0$ and $k > 0$.

2. Electrical model. We have the second case, i.e. when $h < 0$ in (50), in the Froude pendulum near the upper state of equilibrium.

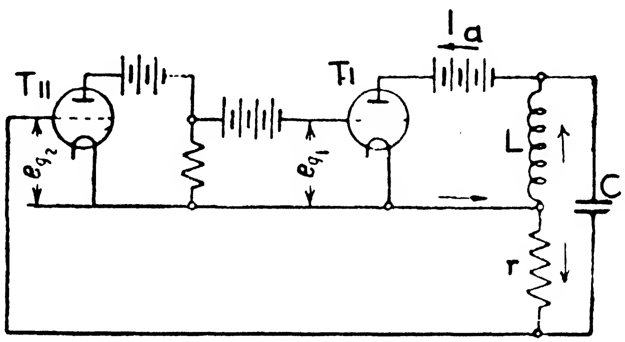

FIG. 40.

Instead of this, we shall consider an electrical system (Fig. 40) which, with a suitable idealization, behaves in the same way.

The scheme of Fig. 40 differs from a vacuum tube oscillator circuit only by the peculiar feedback coupling. In place of the more usual feedback possessing inductance and capacitance, the voltage fed back to the grid is obtained through the resistance r in the plate circuit. The 180° phase shift necessary for the creation of instability (conditions of excitation), is provided by a second tube T_{II} which changes the sign of the input voltage at the grid. In an ordinary scheme with an inductive feedback coupling, the phase shift is achieved by turning the coil of the feedback coupling (by the choice of the sign of M). In the present case, the change of phase cannot be accomplished by such a method, so that a vacuum tube is used. This tube must be linear so as to reverse the phase and increase the voltage without changing its form. If we assume that this condition is fulfilled, it is easy to find the relationship between the grid voltages of the first and the second tube, i.e. between the voltages e_{g_2} and e_{g_1}.

If the directions of the currents are given by the arrows of Fig. 40, we obtain for the grid voltages the following expressions:

$$e_{g_2} = -ri, \qquad e_{g_1} = -ke_{g_2} = kri$$

where k, the amplification of the stage containing T_{II} is a constant depending on the parameters of the vacuum tube T_{II} and the load resistance R in accordance with the relation

$$k = \frac{\mu R}{R + r_p},$$

where μ is the amplification factor of T_{II} and r_p is the plate resistance of T_{II}. Using the established relationship between e_{g_1} and e_{g_2} and applying Kirchhoff's law to the circuit L, C, r, we easily obtain the equation corresponding to our scheme. First of all, neglecting grid currents (in the grid circuit of both tubes), we obtain

$$L\dot{I}_a - L\dot{i} - ri - \frac{1}{C}\int i\,dt = 0.$$

If we neglect the plate voltage, as we have done before, we may assume that

$$I_a = \phi(e_{g_1}) = \phi(kri),$$

so that finally

$$L(kr\phi' - 1)\dot{i} - ri - \frac{1}{C}\int i\,dt = 0.$$

Here ϕ' is $\dfrac{\partial I_a}{\partial e_{g_1}}$, i.e. the mutual conductance of T_I. If we limit ourselves to small values of i, we can (as in the case of the regenerating receiver) consider that the characteristic is linear. Then $\phi' = g_{m_0}$, the mutual conductance characteristic at the operating point, and we obtain

$$L(krg_{m_0} - 1)\dot{i} - ri - \frac{1}{C}\int i\,dt = 0,$$

or, after differentiation with respect to t,

$$L(krg_{m_0} - 1)\ddot{i} - r\dot{i} - \frac{i}{C} = 0.$$

If $krg_{m_0} < 1$, this is an ordinary differential equation of the second order for a system with "attractive force" and positive friction. If $krg_{m_0} > 1$, a condition which can be achieved by a proper choice of k and r, i and $\dot{i}$ have negative coefficients and the equation corresponds

to a system with "repulsive force" and negative friction. As in previous cases, our linear equation can be applied only in a certain limited interval where i is sufficiently small and the characteristics of the tubes can be considered as linear.

Thus the two cases—pendulum in the upper position of equilibrium and vacuum tube generator with resistance feedback—lead to an

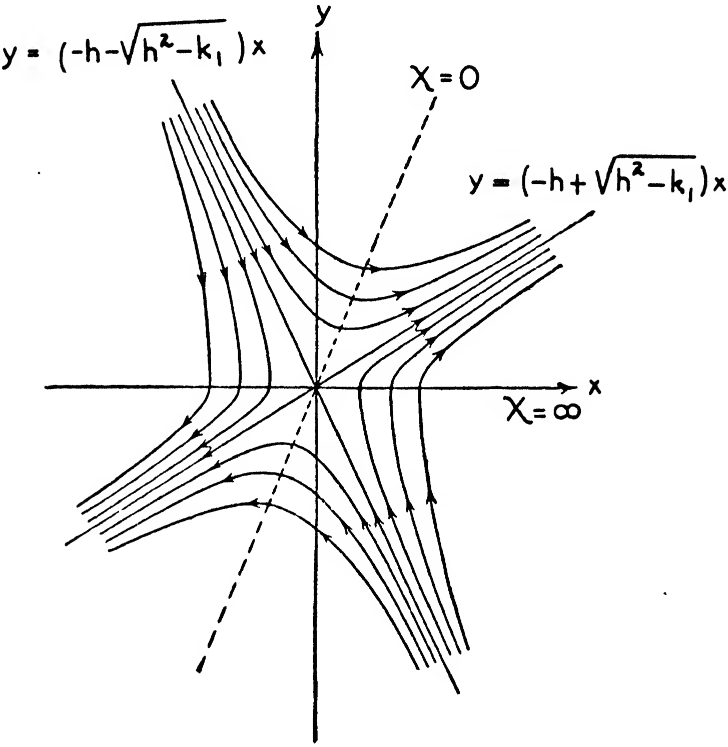

Fig. 41.

equation of the form

$$\ddot{x} + 2h\dot{x} + k_1x = 0, \qquad k_1 < 0$$

with h positive in the first case and negative in the second. The formal treatment of this equation is the same as, for instance, that of (11) and leads in particular to an equation such as (30) for the phase plane trajectories. Here q_1 and q_2 are $h \pm \sqrt{h^2 - k_1}$ and therefore both real, one of them, say q_1, > 0 and the other, q_2, < 0. Let us set $q_1 = q$, $q_2 = -q'$, $q' > 0$. Then (30) assumes the form

$$(y + qx)^q \cdot (y - q'x)^{q'} = C.$$

Hence setting $y + qx = u$, $y - q'x = v$, we have the u,v equation

$$u^q v^{q'} = C$$

or with $r = q'/q > 0$,

$$uv^r = D, \qquad u = \frac{D}{v^r},$$

where D, like C, is an arbitrary constant.

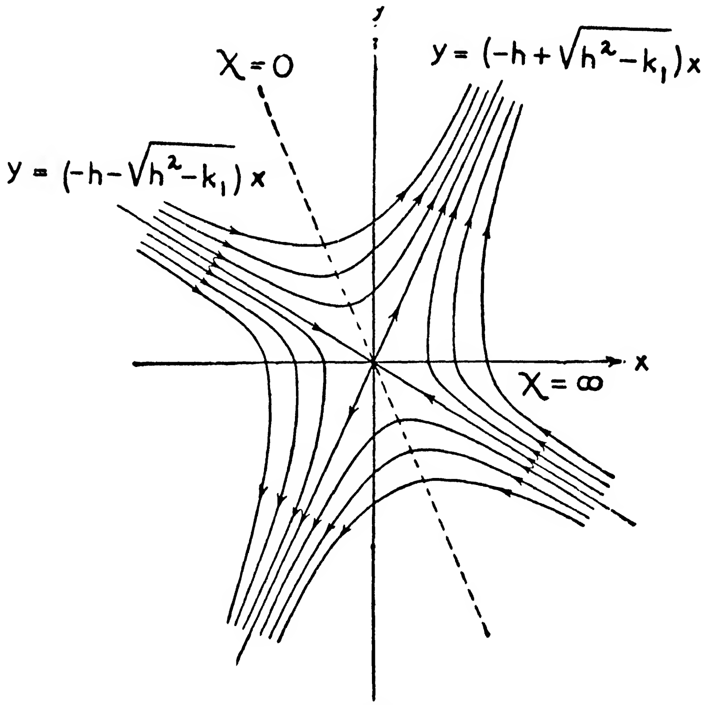

FIG. 42.

In the u,v plane the curves are hyperbolas and hence they are the same in the x,y plane. The discussion is the same as before for a saddle point, and details may safely be omitted, all the more since Figs. 41 and 42, which correspond respectively to h positive and h negative, supply ample information. Thus negative friction does not essentially modify the situation, whose characteristic feature continues to be a saddle point and unstable equilibrium appropriate to such a point.

We terminate our treatment of linear systems with a fundamental observation for the sequel: None of the phase plane diagrams, except

the one corresponding to a harmonic oscillator without friction (i.e. a conservative linear system), represented closed paths, and all the paths had branches leading to infinity. Yet, to the periodic processes there

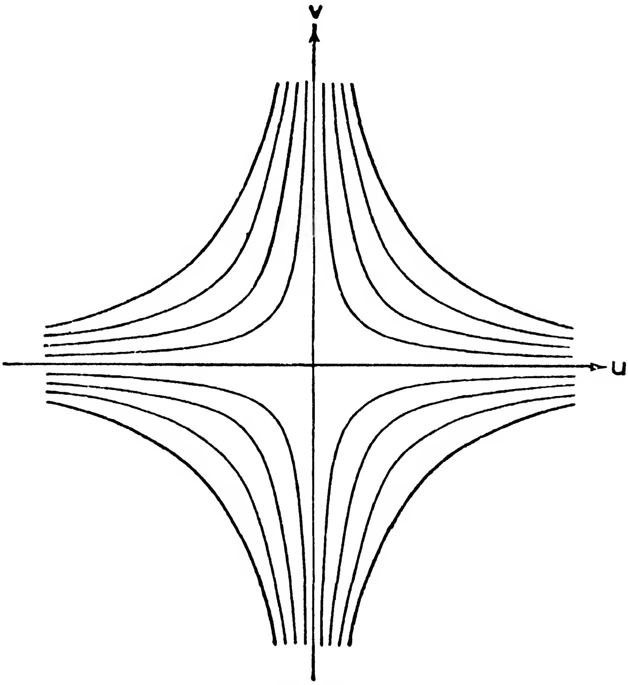

Fig. 43.

must correspond closed paths. Therefore, our investigation of linear systems leads to the following important conclusion: *Periodic processes cannot occur in linear non-conservative systems.*

CHAPTER II

Non-Linear Conservative Systems

§1. Introduction

While dissipation exists in all natural systems, its action is sometimes so slow that its effect may be disregarded. The sum of kinetic and potential energy is then assumed to be constant, i.e. one assumes the law of conservation of energy and the system is regarded as conservative.

To some extent the time during which the system is considered may settle the question. Take for example an almost frictionless pendulum suspended on knife edges in vacuum. One may consider it as forming a conservative system during a short time but not during a long period of time. Similarly the motion of the earth may be taken as conservative throughout many centuries but not throughout geological periods. For the latter it would be necessary to take into account tidal friction, and this would prevent us from regarding the system as conservative.

The study of conservative systems will be of great value to us first for its own sake and also as an occasion for introducing many concepts required later.

§2. The Simplest Conservative System

The simplest conservative system with one degree of freedom consists of a material point in rectilinear motion under the action of a force which depends solely upon the displacement. The position of the point is determined by a single coordinate x, and the mechanical state of the system by x and the velocity $\dot{x}$. To simplify matters, assume the point of mass one. The equation of motion is then

$$\ddot{x} = f(x) \tag{1}$$

where f is the force. An equivalent system is

$$\dot{x} = y, \qquad \dot{y} = f(x). \tag{2}$$

In the sequel we shall generally assume that $f(x)$ is an analytic function for every x, i.e. it may be expanded in Taylor series about every point x.[1] As we have seen, the differential equation defining the paths is

[1] A function $f(x)$ is analytic at the point a whenever it can be expanded in

(3) $$\frac{dy}{dx} = \frac{f(x)}{y} = \phi(x,y).$$

We have shown that, since y is the velocity, then for $y > 0$, i.e. in the upper half-plane, the representative point moves in such manner that x increases and for $y < 0$, i.e. in the lower half-plane, it moves so that x decreases. The *direction* of the motion over the phase plane is thus defined. The speed of motion of the representative point is

(4) $$|v| = \dot{s} = \sqrt{\dot{x}^2 + \dot{y}^2} = \sqrt{y^2 + f^2(x)}.$$

Let us recall that one should distinguish the velocity of the material point from the velocity of the motion of the representative point on the phase plane. The first velocity $\dot{x} = y$ is equal to the ordinate, while the magnitude of the second

$$\dot{s} = \sqrt{y^2 + f^2(x)} = |y|\sqrt{1 + \left(\frac{dy}{dx}\right)^2}$$

is equal to the length of the normal to the path at the representative point. The expression (4) shows that the representative point has a finite non-zero velocity at all points of the phase-plane with the exception of the positions of equilibrium (singular points) in which $y = 0$, $f(x) = 0$.

Thus all the equilibrium positions correspond to points on the x-axis, and in fact to the solutions of $f(x) = 0$.

Through every point (x_0,y_0) of the phase plane where the conditions of Cauchy's existence theorem are fulfilled[1] by (3), there passes a

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a Taylor series about a (MacLaurin series for $a = 0$), i.e. if

$$f(x) = f(a) + (x - a)f'(a) + \frac{(x - a)^2}{2!}f''(a) + \cdots$$

where the series is convergent for $|x - a|$ sufficiently small. The function $f(x)$ is holomorphic in an interval $b < x < c$, or $-\infty < x < +\infty$, if it is analytic at each point of the interval. Similarly $f(x,y)$ is analytic at the point $M(a,b)$ if it can be expanded in a Taylor series about M (in powers of $x - a$, $y - b$) convergent for $|x - a|$ and $|y - b|$ sufficiently small. It is holomorphic in a region R if it is analytic and single-valued at each point of R. The extension to any number of variables is obvious.

[1] Cauchy's theorem for $dy/dx = \phi(x,y)$ asserts that whenever ϕ is analytic at $M(a,b)$ there exists a unique solution $y(x)$ analytic at M and such that $y(a) = b$. Moreover, if $\phi(x,y)$ is holomorphic in a region R, the solution $y(x)$ can be extended throughout R. Roughly speaking, the singular points are those where ϕ ceases to be analytic; thus for (3) the points where $y = 0$ are singular. Similarly for a pair

unique path, a solution of (3). If we consider y as a function of x defined by (3), and since $\phi_y = -f(x)/y^2$, Cauchy's conditions do not hold on the x-axis ($y = 0$). On the other hand, consider now x as a function of y defined by $dx/dy = y/f(x)$. This time

$$\phi_x = \frac{-yf'(x)}{f^2(x)},$$

and so Cauchy's conditions fail now on the lines $f(x) = 0$. Since the paths remain the same, there will pass one and only one through any point but those where $y = 0, f(x) = 0$, i.e. through every non-singular point. Referring to (4) we see that the velocity of the representative point tends to zero as it tends to a singular point.

While the paths are represented directly by (3) and only parametrically by (2), there is a well-known difference as regards the singular points. For instance, let the origin be such a point. Under our assumptions (2) satisfies the Cauchy conditions at the origin and so $x(t) = 0$, $y(t) = 0$ is a "point" solution of (2) and it is the unique solution of (2) "through" the origin. No other path as defined by means of (2) can reach the origin. Now let (3) yield a path passing through the origin. Outside the origin and near the origin this curve will also satisfy (2). However, if the representative point describing the curve tends to the origin, it cannot reach it in finite time since otherwise (2) would have more than one solution through the origin. Instances of this have already been found in Chap. I, notably in connection with stable nodes.

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of equations

$$(\alpha) \qquad \frac{dx}{dt} = \phi(x,y,t), \qquad \frac{dy}{dt} = \psi(x,y,t),$$

where ϕ, ψ are analytic at $M(a,b,t_0)$ or holomorphic in a region R. We have then a unique analytic solution $x(t),y(t)$ (a unique vector) such that $x(t_0) = a$, $y(t_0) = b$. The solution represents a unique curve through M. For an equation of order two

$$(\beta) \qquad \ddot{x} + f(x,\dot{x},t) = 0$$

we proceed as follows. It is supposed that $f(x,y,t)$ is analytic at $(x_0,\dot{x}_0,t_0)$. Setting $\dot{x} = y$, we find the system

$$(\gamma) \qquad \dot{x} = y, \qquad \dot{y} = -f(x,y,t)$$

with right-hand sides analytic at the same point. Thus (γ) is of the same nature as (α). Hence it has a unique analytic solution $x(t),y(t)$ such that $x(t_0) = x_0$, $y(t_0) = \dot{x}_0$. Since $y = \dot{x}$, (β) has a unique analytic solution $x(t)$ such that $x(t_0) = x_0$, $\dot{x}(t_0) = \dot{x}_0$, i.e. a unique solution $x(t)$ with preassigned position x_0 and velocity $\dot{x}_0$ at time t_0.

§3. The Phase Plane in the Neighborhood of the Singular Points

Let us introduce the function $V(x)$ given by

$$-V(x) = \int_{x_0}^{x} f(x)\, dx.$$

Thus $-V(x)$ is the work done by $f(x)$ and so $V(x)$ is the potential

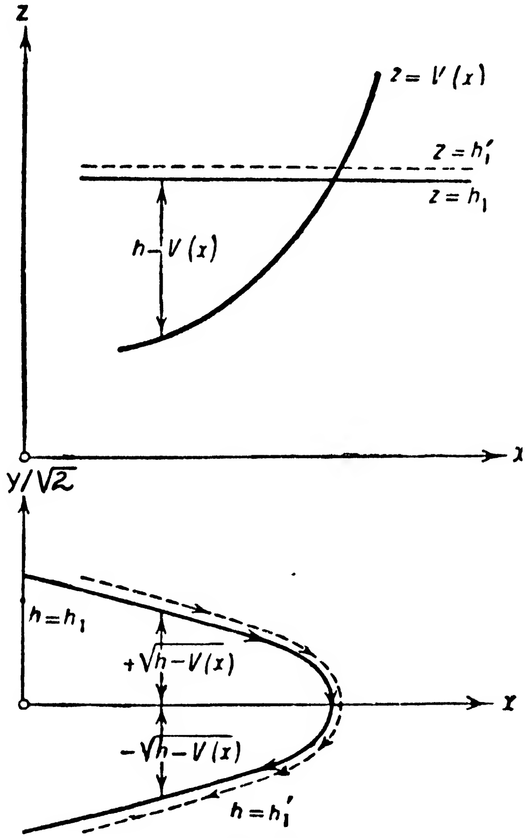


FIG. 44.

energy. The kinetic energy is $\frac{1}{2}\dot{x}^2 = \frac{1}{2}y^2$. Now (1) may be written

$$\ddot{x} + V'(x) = 0.$$

Hence multiplying by $\dot{x}$ and integrating we find

$$F(x,y) = \tfrac{1}{2}y^2 + V(x) = h \tag{5}$$

which reads: kinetic energy + potential energy = constant. In other words, (5) expresses the law of conservation of energy. If x_0,y_0

are the values of x,y at time $t = t_0$, we have

$$\tfrac{1}{2}y_0^2 + V(x_0) = h,$$

so that h is determined by the initial conditions. The curves (5) are the integral curves of (3), and one and only one passes through every point. It is not certain, however, that h may assume all possible values. Moreover, a given curve (5) may consist of several distinct branches, each representing a distinct motion. This would be the case, for instance, if the curves were hyperbolas.

The following properties of the paths of (2) may be pointed out:

(a) Since (5) is unchanged when y is replaced by $-y$, the paths are symmetrical with respect to the x-axis.

(b) The locus of the points where the tangents to the curves are vertical is the x-axis.

(c) The locus of the points where the tangents are horizontal consists of the lines $f(x) = 0$, i.e. of the verticals through the singular points.

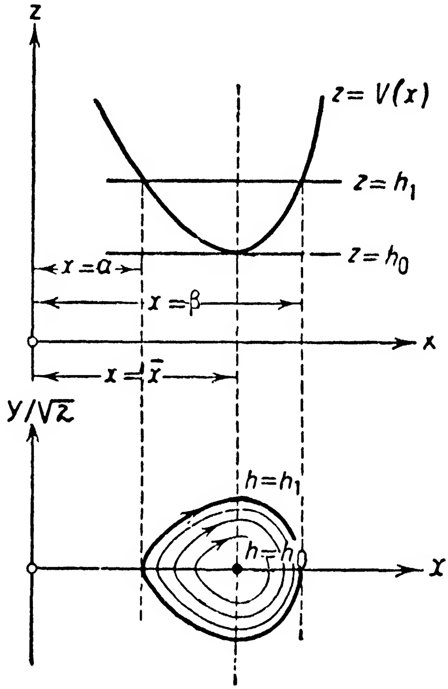


FIG. 45.

Our present paths may be constructed as follows: Corresponding to two horizontal x-axes with origins on the same vertical erect a z-axis and then a vertical axis on which are to be plotted the values of $y/\sqrt{2}$. Draw first the graph $z = V(x)$ and the lines $z = h$, then read off for any x the value $h - z$ and plot $y/\sqrt{2} = \pm\sqrt{h - z}$ in the obvious way. Two particular cases are illustrated in Figs. 44 and 45.

Let $\bar{x}$ be a singular point and set $V(\bar{x}) = h_0$. We propose to examine the general aspect of the paths near the singular point. As a preliminary step we require the expansions of $f(x)$ and $V(x)$ about $\bar{x}$:

$$f(x) = a_1(x - \bar{x}) + \frac{a_2}{2!}(x - \bar{x})^2 + \frac{a_3}{3!}(x - \bar{x})^3 + \cdots, \tag{6}$$

$$V(x) = h_0 - \left\{\frac{a_1}{2!}(x - \bar{x})^2 + \frac{a_2}{3!}(x - \bar{x})^3 + \frac{a_3}{4!}(x - \bar{x})^4 + \cdots\right\}, \tag{7}$$

$$a_1 = f'(\bar{x}) = -V''(\bar{x}), \qquad a_2 = f''(\bar{x}) = -V'''(\bar{x}), \qquad \cdots.$$

The expansion (6) of $f(x)$ is merely its Taylor series about $\bar{x}$. The first term is absent since $f(\bar{x}) = 0$. The second series represents the Taylor expansion of $V(x)$ where $h_0 = V(\bar{x})$ and the coefficients are deduced from those of $f(x)$ together with $f = -V'$. They should be $V'(\bar{x}) = -f(\bar{x}), \cdots$, but $V'(\bar{x}) = 0$ at the extremum $\bar{x}$ and so there is no first degree term in (7).

Making now the transformation of coordinates $x = \bar{x} + \xi$, $y = 0 + \eta = \eta$, the equation (5) of the paths becomes

$$\frac{\eta^2}{2} + h_0 - \left\{\frac{a_1\xi^2}{2!} + \cdots + \frac{a_k\xi^{k+1}}{k!} + \cdots\right\} = h. \tag{8}$$

We will first suppose $a_1 = -V''(\bar{x}) \neq 0$. Confining our attention to the vicinity of the singular point, i.e., of $\xi = \eta = 0$ (the origin in the ξ,η coordinates), we may neglect the powers of ξ greater than two and replace (8) by

$$\eta^2 - a_1\xi^2 = 2(h - h_0). \tag{9}$$

Thus as h varies we have a family of conics whose nature depends on the sign of a_1, i.e. of $V''(\bar{x})$. There are three cases accordingly as a_1 is negative, positive, or zero. They correspond respectively to a potential energy minimum, maximum, or "in between" at the singular point.

Consider the first case: $a_1 < 0$ and hence the potential energy is a minimum. We can only take $h \geqslant h_0$. Setting $a^2 = 2(h - h_0)$, $b^2 = 2(h - h_0)/-a_1$, (9) becomes

$$\frac{\eta^2}{a^2} + \frac{\xi^2}{b^2} = 1.$$

The approximate paths are similar concentric ellipses and the true paths are concentric ovals around the singular point (Fig. 45). This may also be deduced directly from the construction of the curves for h slightly larger than h_0. The qualitative (topological) aspect of the scheme is that of a center. The singular point is thus stable.

It may be noticed that, since h_0 is a minimum of $V(x)$, the line $z = h_0$ is tangent to the graph $z = V(x)$ at the minimum point. By reference to the system (2) we verify that the ovals are described as indicated in Fig. 45.

A closely related case, corresponding again to a minimum of $V(x)$, is the following: $a_1 = \cdots = a_{2k-2} = 0$, $a_{2k-1} < 0$. Neglecting now

the powers of ξ greater than $2k$ we have in place of (9) the equation

$$\frac{\eta^2}{2} - \frac{a_{2k-1}\xi^{2k}}{(2k)!} = (h - h_0).$$

Setting $-2a_{2k-1}/(2k)! = \alpha^2$, $2(h - h_0) = \beta^2$, the equation may be reduced to

$$\eta^2 = \beta^2 - \alpha^2\xi^{2k}.$$

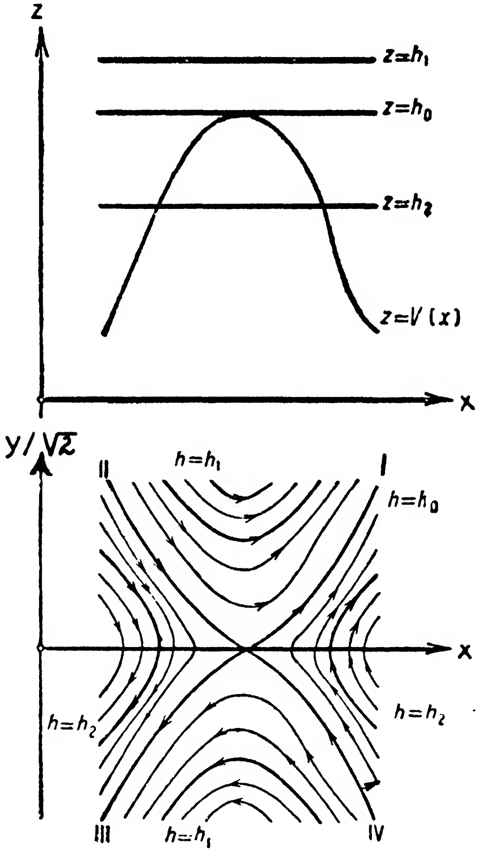


FIG. 46.

By plotting

$$\eta = \pm\sqrt{\beta^2 - \alpha^2\xi^{2k}}$$

one sees that near the singular point we have again concentric ovals. This can only occur, of course, with the general aspect of Fig. 45. In other words, the special case just considered does not differ from the previous one.

The next general case is $a_1 > 0$ and hence the potential energy

is maximum at the singular point. It is not necessary to repeat the discussion. The approximate curves are now

$$\eta^2 - a_1\xi^2 = 2\alpha, \qquad \alpha = h - h_0,$$

and as h, hence α varies, we obtain a family of hyperbolas symmetrical with respect to the axes and with the fixed asymptotes

$$\eta = \pm \sqrt{a_1}\,\xi.$$

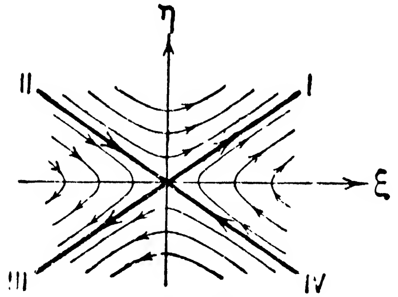
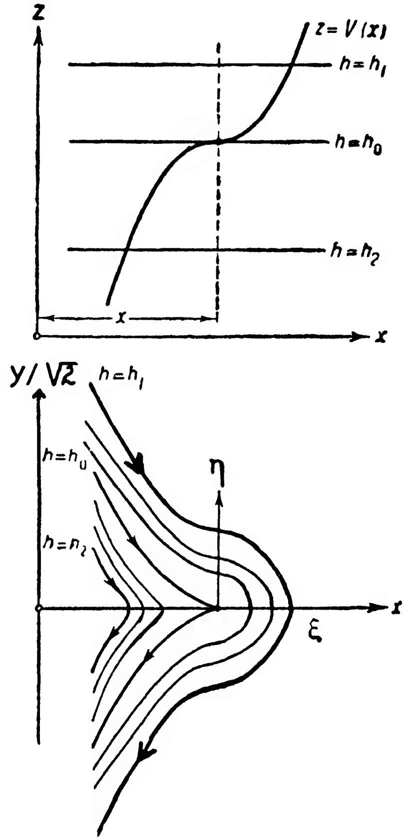


FIG. 47.



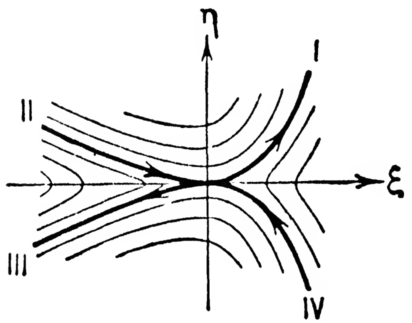


FIG. 48.

FIG. 49.

They are indicated in Fig. 47. The exact curves are those of Fig. 46. The singular point is essentially a saddle point. The exceptional branches I, II, III, IV of Fig. 46 correspond to $h = h_0$. The diagrams clearly indicate what is happening.

Here again $a_1 = \cdots = a_{2k-2} = 0$, $a_{2k-1} > 0$, corresponding to a maximum of $V(x)$, yields a variation of the above situation. In the ξ, η-plane the equation is

$$\eta^2 = \pm\beta^2 + \alpha^2\xi^{2k}$$

where α^2 is fixed and β^2 varies with h. By construction this is easily shown to yield Fig. 48 and it still corresponds essentially to a saddle point.

There remains the intermediary case of a horizontal inflection (Fig. 49). This time we have $a_1 = \cdots = a_{2k-1} = 0$, $a_{2k} \neq 0$ and the approximate equation is

$$\frac{\eta^2}{2} - \frac{a_{2k}\xi^{2k+1}}{(2k+1)!} = h - h_0$$

which may be sufficiently understood by reference to Fig. 49. In character we have here a sort of degenerate saddle point and instability.

From the preceding discussion one may infer that when the potential energy of the system possesses a minimum, the equilibrium is stable, while if it has a maximum or a horizontal inflection the equilibrium is unstable. This proves for our conservative system two fundamental stability theorems: the theorem of Lagrange which reads,

If the potential energy is minimum at the state of equilibrium, the equilibrium is stable;

and the converse theorem of Liapounoff,

If the potential energy is not a minimum at the state of equilibrium, then the equilibrium is unstable.

Let us look now for the multiple points of the paths (5). They are the points where $F_x = F_y = 0$. Since

$$F_x = V'(x), \qquad F_y = y,$$

the multiple points are the solutions of

$$-V'(x) = f(x) = 0, \qquad y = 0$$

and so we see that they are merely the singular points of the system.

§4. Discussion of the Motion in the Whole Phase Plane

Our general construction by means of the auxiliary curve $z = V(x)$ may be applied equally well to a complete motion. The only novel element not yet discussed is the presence of possible infinite branches in the path. They will only occur if $z = V(x)$ has infinite branches below the line $z = h$, as is seen by reference to the figures. If one suppresses the parts of the curve above $z = h$, the remaining part will in general contain disconnected arcs. Thus in Fig. 51 there are three: an infinite branch (lower left) and two finite arcs. The infinite branch yields an infinite branch of the path going from and back to infinity,

and each finite arc gives rise to two isolated ovals. If the initial point is on one of the three, it remains on it. Thus, corresponding to h and depending upon the initial values of $x, \dot{x}$, there may occur here either an aperiodic motion (finite or infinite arc) or one of two periodic motions (ovals).

A closed path occurs then whenever there exists an arc of the xz-curve joining two points of the line $z = h$ and lying below the line.

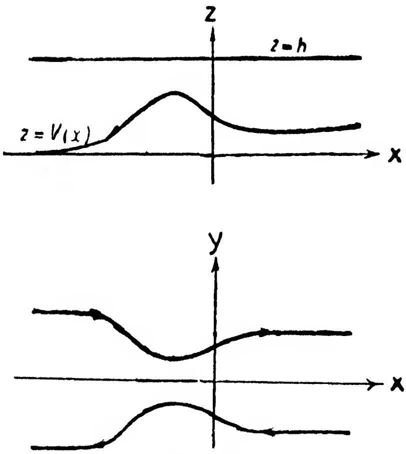


FIG. 50.

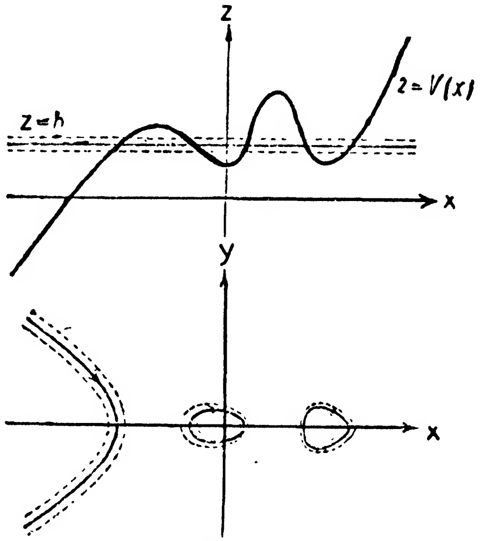


FIG. 51.

On such an arc, for instance the one at the right in Fig. 52, the slope $V'(x)$ is negative at one end and positive at the other. Hence V' will change sign an odd number of times and pass one more time from $-$ to $+$ than from $+$ to $-$ on the arc. In other words: number of minima $-$ number of maxima $= 1$ on the arc. Hence the following theorem due to Poincaré:

In a conservative system, the singular points interior to a closed path are saddle points and centers. Their total number is odd and the number of centers exceeds the number of saddle points by one.

A *separatrix* is a path γ tending to a singular point A (generally a saddle point) as $t \to +\infty$ (or $-\infty$), such that the neighboring paths do not tend to A under the same conditions and so part from γ as $t \to +\infty$ (or $-\infty$). The separatrices are the curves indicated by heavy lines in the figures. Since a representative point can never cross a singular point, each arc of separatrix between singular points represents a motion on which the point tends to a singular point, or

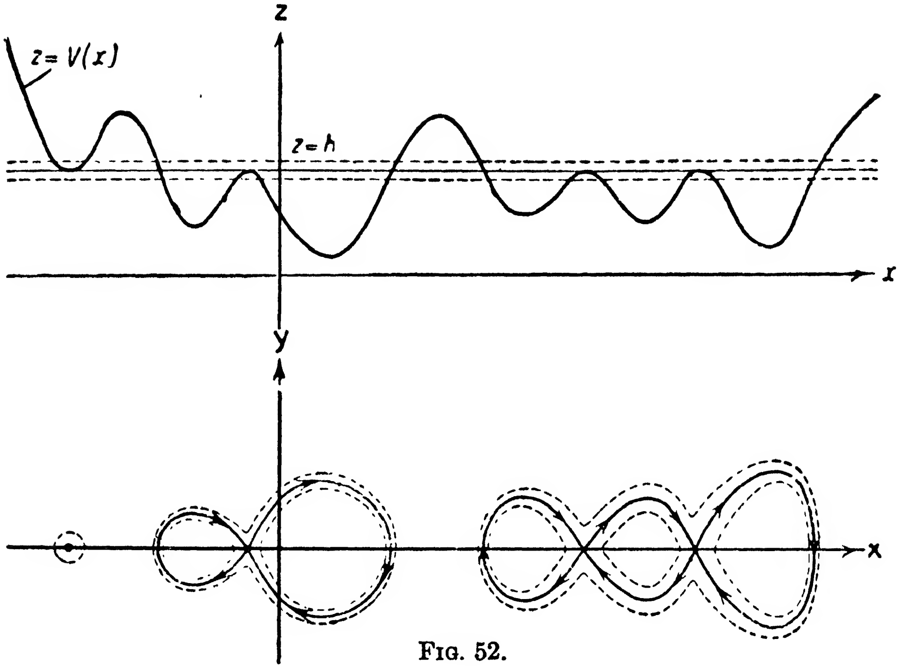


Fig. 52.

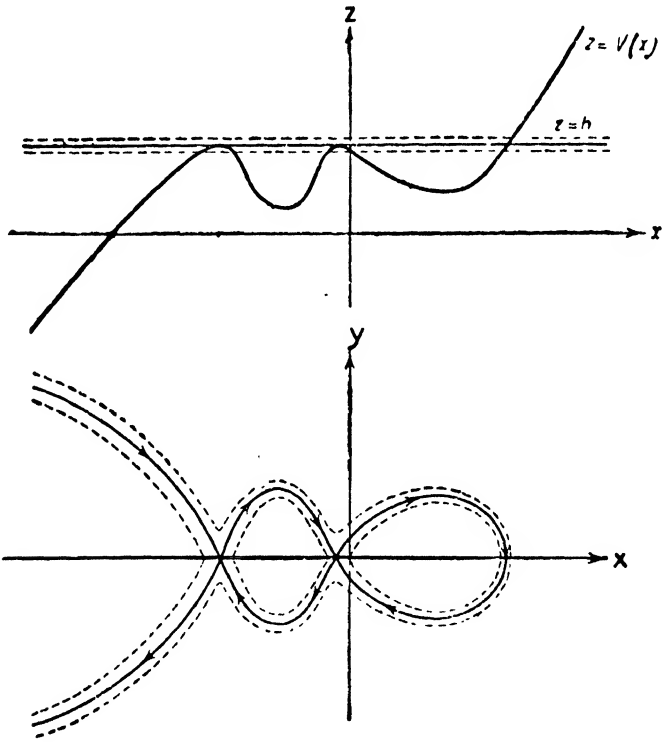


Fig. 53.

the system to a position of equilibrium, in infinite time. Thus in Fig. 53 there are five such arcs, one of them, at the extreme right, going from the last saddle point at the right back to that point.

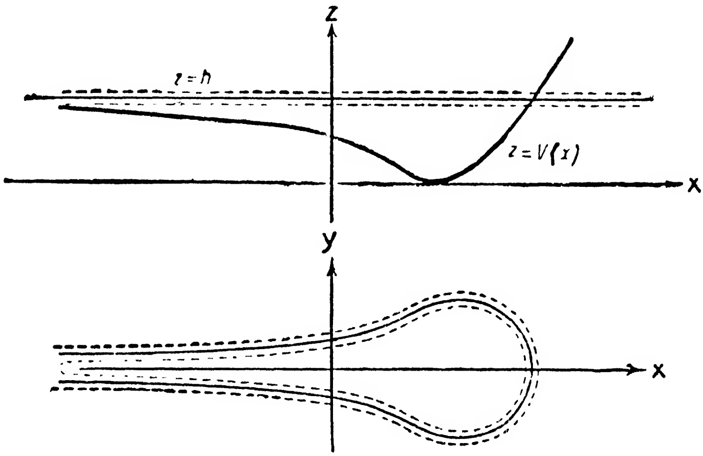


FIG. 54.

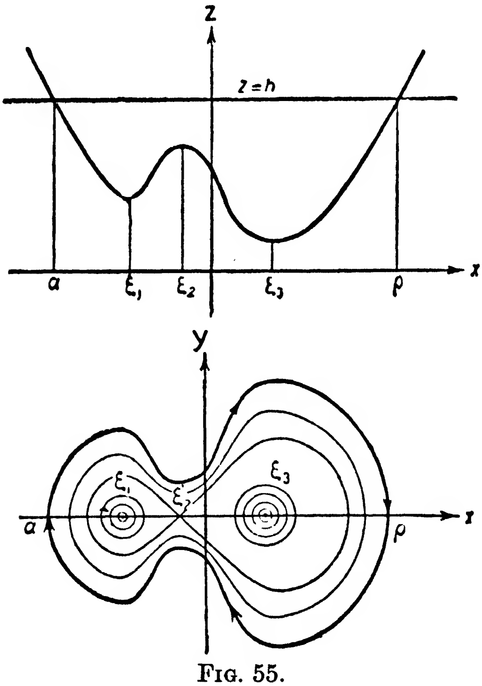


FIG. 55.

Let h_1 be the value of h corresponding to a separatrix. When h increases above h_1, the path surrounds the entire separatrix—for example the exterior dotted lines in Fig. 52. When h decreases,

there arise closed paths exterior or entirely interior to the separatrix and each around a center. Thus in Fig. 52 at the left there arise three, two interior to the separatrix and one outside at the extreme left. The last one is caused by the presence of an isolated point of the separatrix. Notice also the behavior described in Fig. 55.

§5. Dependence of the Behavior of a Conservative System upon Certain Parameters

Let us suppose that the force f is a function $f(x,\lambda)$ analytical for all values of x and λ, that is to say possessing a Taylor expansion for all x,λ. The behavior of the system may change with λ. Roughly

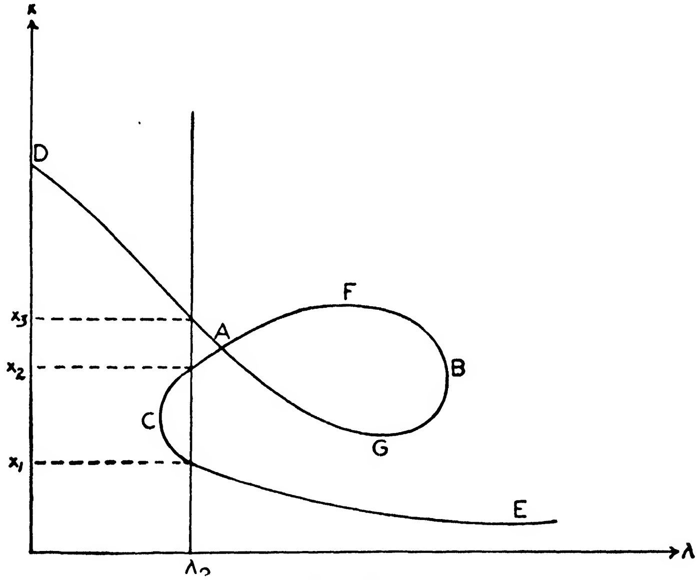


Fig. 56.

speaking, one may say that the changes are merely quantitative, unless λ crosses certain values called *branch points* (*bifurcations* in Poincaré's terminology) where the very nature of the system is modified. To be more precise, let us call λ_0 an *ordinary* value of λ whenever there exists an $\epsilon > 0$ such that for any λ such that $|\lambda - \lambda_0| < \epsilon$ the topological (i.e. qualitative) nature of the paths is the same as for λ_0. When λ_0 does not behave in this manner, it is said to be a branch point.

Let us discuss, following Poincaré, what happens when λ varies. The positions of equilibrium are given by the roots $x_1, \cdots, x_s$ of $f(x,\lambda) = 0$ in x. On the graph of Fig. 56 those corresponding to λ_0 are the points of the graph on the line $\lambda = \lambda_0$.

Let $\bar{x}$ designate any one of the positions of equilibrium x_i. Since $V_{xx}(x,\lambda) = -f_x(x,\lambda)$, $\bar{x}$ will be a minimum whenever $f_x(\bar{x},\lambda) < 0$ and a maximum whenever $f_x(\bar{x},\lambda) > 0$. Assuming that there are no cusps, the number of singular points x_i for given λ, may increase or decrease by two units and this when and only when λ crosses a value λ_0 where

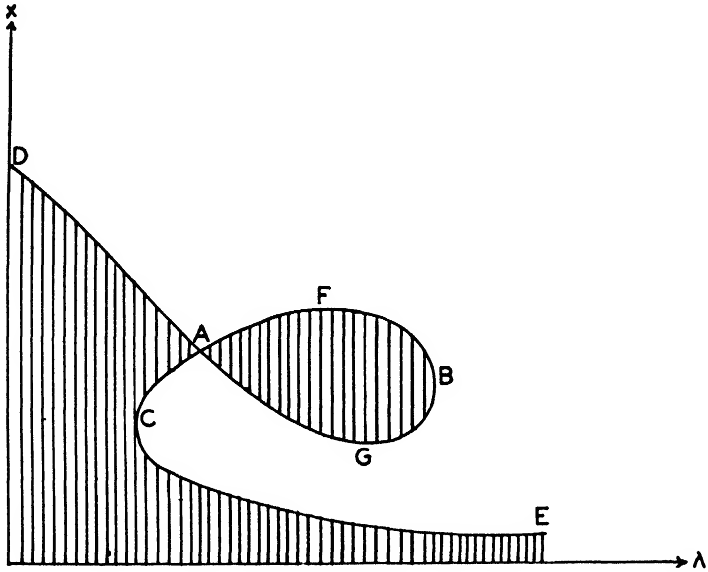


FIG. 57.

$\lambda = \lambda_0$ is tangent, i.e. where

$$f_x(\bar{x},\lambda_0) = 0, \qquad f_\lambda(\bar{x},\lambda_0) \neq 0.$$

For the slope at any point (x,λ) of the graph is

$$\frac{dx}{d\lambda} = -\frac{f_\lambda}{f_x}$$

and under the above conditions it is infinite at $(\bar{x},\lambda_0)$, hence the tangent there is $\lambda = \lambda_0$. Thus λ_0 is a branch point. The discussion of what happens at the multiple points of $f(x,\lambda) = 0$ may of course be carried quite far, but this need not be done here.

There is a simple rule due to Poincaré for determining rapidly the parts of the x,λ-graph corresponding to stability or instability. Let the shaded region (Fig. 57) consist of the points where $f(x,\lambda) > 0$. If the point $(\bar{x},\lambda)$ of the graph $f(x,\lambda)$ is on an arc γ of the graph *above* the region, then $\bar{x}$ represents a position of stable equilibrium; on the contrary, if γ is below the region, we have unstable equilibrium. The

reason is quite simple. In the first case, for λ fixed and x increasing, $f(x,\lambda)$ decreases as x crosses $\bar{x}$, hence $f_x(\bar{x},\lambda) < 0$ or we have a center and stability. In the second case, $f(x,\lambda)$ increases through $\bar{x}$, $f_x(\bar{x},\lambda) > 0$ and so we have a saddle point and instability. Thus, according to this rule, the arcs DA, AFB, CE correspond to stability and the arcs AGB, AC to instability.

Upon describing an arc of the graph, i.e. upon varying λ continuously, the stability properties are unchanged until a branch point is crossed. There a stable and unstable position of equilibrium are permuted. As λ crosses the branch point λ_0, stable and unstable positions appear or disappear in pairs.

To illustrate the preceding theory we shall discuss a few examples.

1. Motion of a material point on a circle which rotates about a vertical axis. Consider the motion of a material point of mass m on a circle of radius a rotating about a vertical diametral axis with constant angular velocity Ω (Fig. 58). A pendulum fixed on a pivot which rotates around a vertical axis is a model of such a system.

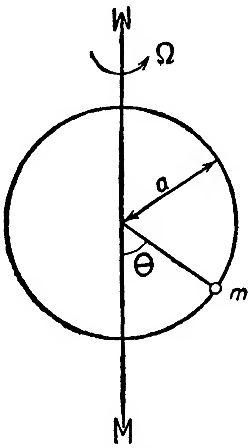


FIG. 58.

In order to study the motion along the angular coordinate θ it is convenient to introduce a coordinate system revolving with the pendulum around the vertical axis. To apply Newton's second law relative to the moving axes, one must introduce the centrifugal force. The moment of the force of gravity with respect to the horizontal axis is $mga \sin \theta$. The centrifugal force is $m\Omega^2 a \sin \theta$ and its moment with respect to the horizontal axis is $m\Omega^2 a^2 \sin \theta \cos \theta$. Since the two moments are directed oppositely, the resulting moment is $m\Omega^2 a^2(\cos \theta - \lambda) \sin \theta$ where $\lambda = g/\Omega^2 a$. Let us note that $\lambda = 1$ when $\Omega = \sqrt{g/a}$, i.e. when the angular velocity coincides with the angular frequency of small oscillations of the point m as a pendulum (for $\Omega = 0$). We shall investigate the behavior of the system for different λ, i.e. for different values of the angular velocity Ω and g assuming that g can change not only in value but also in sign, i.e. that λ may be negative. If I is the moment of inertia of the system with respect to the horizontal axis, the equation of motion is

$$I \frac{d\omega}{dt} = m\Omega^2 a^2(\cos \theta - \lambda) \sin \theta, \qquad \frac{d\theta}{dt} = \omega.$$

The equation of the paths on the phase plane (of θ and ω) is

$$\frac{d\omega}{d\theta} = \frac{m\Omega^2a^2(\cos\theta - \lambda)\sin\theta}{I\omega}.$$

The energy integral is

$$\frac{I\omega^2}{2} - m\Omega^2a^2\left(\frac{\sin^2\theta}{2} + \lambda\cos\theta\right) = C. \tag{10}$$

Consider now the behavior of the paths on the phase plane θ,ω. The singular points (positions of equilibrium) are $\omega_1 = 0$, $\theta_1 = 0$; $\omega_2 = 0$,

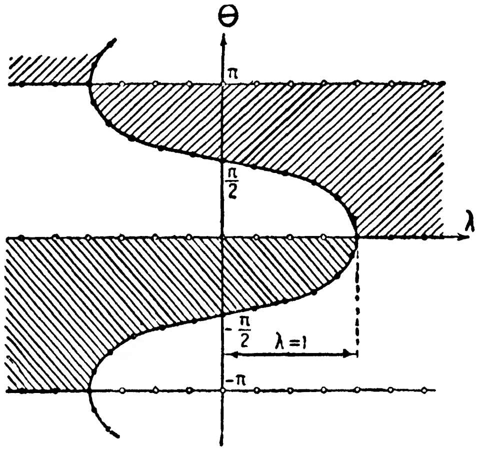


FIG. 59.

$\theta_2 = \pi$; $\omega_3 = 0$, $\cos\theta_3 = \lambda$. The last equilibrium position exists only when $|\lambda| \leqslant 1$, i.e. when the frequency of the small oscillations of the pendulum (for $\Omega = 0$) is smaller than the rotational frequency. The coordinate θ of the position of equilibrium is given by

$$f(\theta,\lambda) = m\Omega^2a^2(\cos\theta - \lambda)\sin\theta = 0.$$

The graph of this relation consists of the curve $\cos\theta = \lambda$ and of the straight lines $\theta = 0$, $\theta = \pi$, $\theta = -\pi$, etc. (Fig. 59). It is obvious that for $0 < \theta < \pi$ the region where $f(\theta,\lambda) < 0$ is on the side of increasing λ, and for $0 > \theta > -\pi$ on the side of decreasing λ (in Fig. 59 these regions are shaded). The stable (black points) and unstable (white points) states of equilibrium may now be determined by the rule given before. Since λ may be negative or positive, the singular points for which $\theta = \pm\pi$ are unstable (saddle points) for $\lambda > -1$ and stable (centers) for $\lambda < -1$. The singular points for which $\theta = 0$ will be

unstable for $\lambda < 1$ and stable for $\lambda > 1$. Finally the singular points for which $\lambda = \cos\theta$ are always stable but exist only if $-1 < \lambda < 1$. As one could expect, the picture for θ is completely symmetrical, and the states corresponding to $+\theta$ and $-\theta$ coincide. We can determine the separatrices remembering that each passes through a singular point in which the energy constant can be calculated: the kinetic energy is equal to zero in the singular point (the system is at rest) and it is easy to calculate the potential energy. Substituting the value of the energy

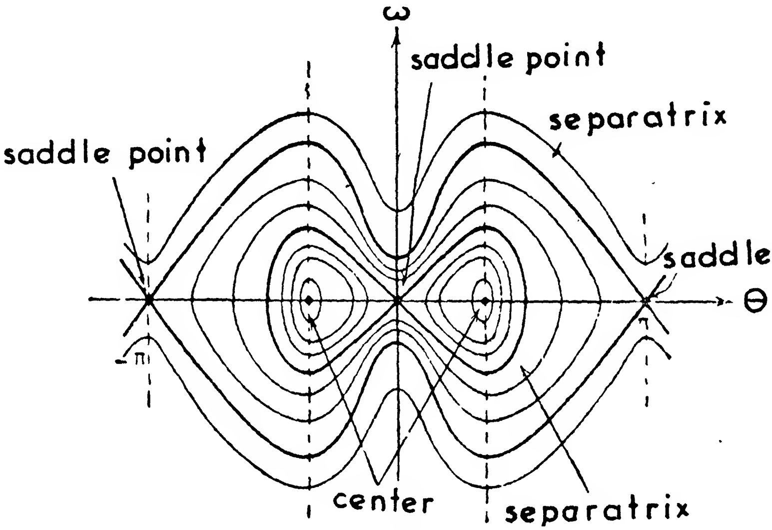


FIG. 60.

constant in (10), we obtain two separatrices forming a figure eight and passing through a saddle point. The equation of one of them, passing through the point $\omega = 0$, $\theta = \pi$, where $C = mga$, is

$$\omega^2 = \frac{m\Omega^2a^2}{I}(\sin^2\theta + 2\lambda(\cos\theta + 1)).$$

The equation of the second, passing through the saddle point $\omega = 0$, $\theta = 0$, where $C = -mga$, is

$$\omega^2 = \frac{m\Omega^2a^2}{I}(\sin^2\theta + 2\lambda(\cos\theta - 1)).$$

Both are represented in Fig. 60 for $1 > \lambda > 0$. When $\lambda = 0$ the two separatrices merge and we obtain the situation of Fig. 61. When $-1 < \lambda < 0$, the situation is the same as for $0 < \lambda < 1$, with the difference that in the second case the figure is displaced along the

θ-axis by π (Fig. 62). When $0 < \lambda < 1$ (Fig. 60), there are three regions of periodic motions, inside the first separatrix. Two of these regions are simply connected and one doubly connected. Since the separatrices merge for $\lambda = 0$, the doubly connected region disappears.

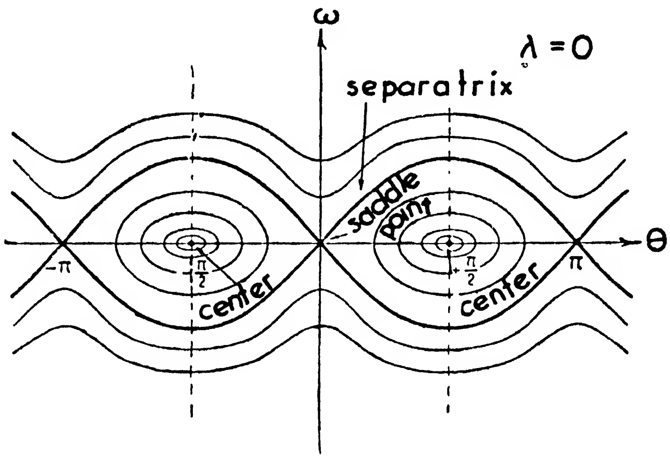


Fig. 61.

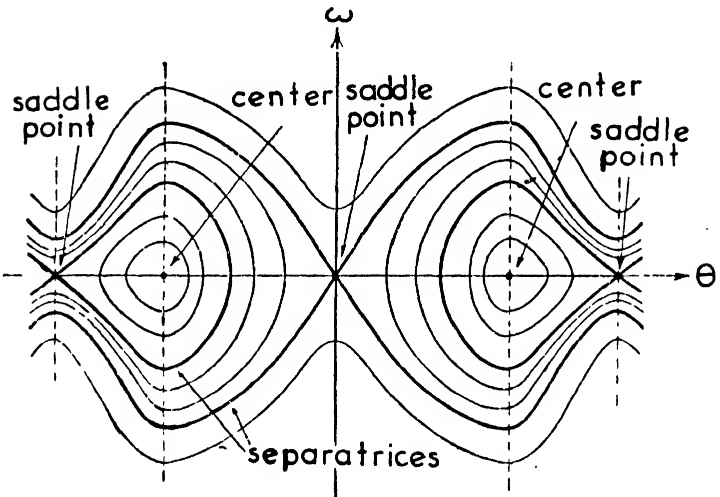


Fig. 62.

The phase portrait changes and brings out the significance of the branch point of the parameter for the separatrices. In a similar way, when $|\lambda| > 1$, we obtain a new portrait (Fig. 63) and hence the values $\lambda = \pm 1$ are also branch points.

2. Motion of a material point on a parabola rotating around a vertical axis. This example is like the preceding save that the

circle is replaced by the parabola with vertical axis $x^2 = 2pz$, which undergoes a rotation with constant angular velocity Ω around the axis (Fig. 64). The potential energy is the energy of the mass in the field of gravity or

$$V = mgz = mg\frac{x^2}{2p}.$$

The kinetic energy T is the sum of the energy of rotation around the

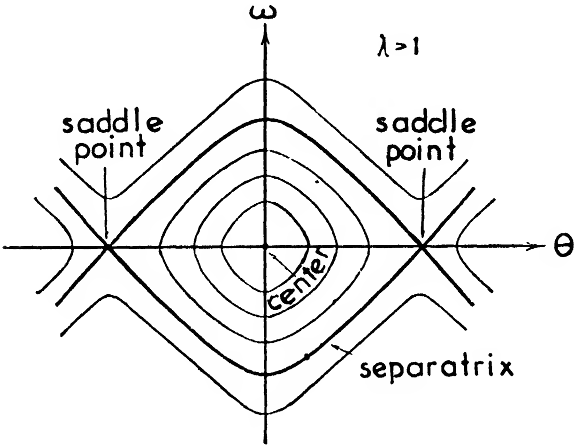


FIG. 63.

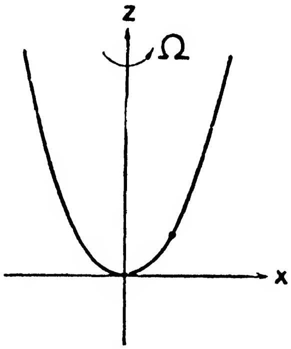


FIG. 64.

vertical axis and of the energy of motion on the xz-plane, or

$$T = \frac{m\Omega^2x^2}{2} + \frac{m}{2}(\dot{x}^2 + \dot{z}^2).$$

From

$$x\dot{x} = p\dot{z}, \qquad \dot{x} = y,$$

and an application of Lagrange's equation we deduce then the equation of the paths in the phase plane x,y:

$$\left(1 + \frac{x^2}{p^2}\right)y^2 + \lambda x^2 = C.$$

Their singular points occur at $y = 0$, $\lambda x = 0$. The analysis of the possible types of motion according to the values of λ follows:

$\lambda > 0(\Omega^2 < g/p)$. The origin is a center so that the equilibrium is stable. The paths are concentric ovals around the origin (Fig. 65).

$\lambda = 0(\Omega^2 = g/p)$. There is an unlimited number of states of equilibrium corresponding to the line $y = 0$. The paths are represented in Fig. 66. The point will either be at rest on the parabola

or will move monotonically in the direction of the initial velocity. When $t \to +\infty$, the velocity $\to 0$; it is maximal at the vertex of the parabola.

$\lambda < 0 (\Omega^2 > g/p)$. There is one saddle point at the origin; the straight lines $y = \pm \sqrt{-\lambda} \cdot p$ are paths, and correspond to the motion

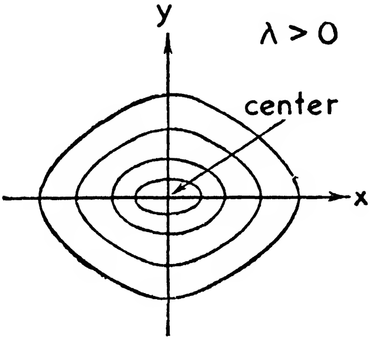


Fig. 65.

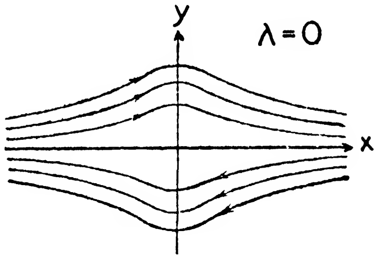


Fig. 66.

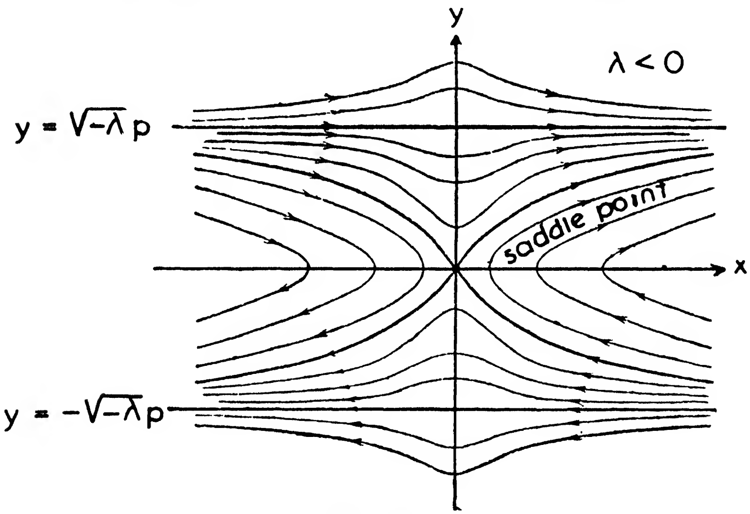


Fig. 67.

of the point with constant velocity on the parabola. If the initial velocity exceeds $\sqrt{-\lambda} \cdot p$, the motion is like that for $\lambda = 0$; from lower initial velocities the point either moves monotonically in one direction with minimum velocity at the vertex of the curve, or else it returns without reaching the vertex.

3. A special rectilinear motion. Let a material point on a straight line referred to the coordinate x be subjected to a force of the form

$$F = \mp \frac{\alpha}{x^2} + \frac{\gamma}{x^3}, \quad \alpha \text{ and } \gamma \geq 0,$$

where the sign is that of $-x$. Thus $\mp \alpha/x^2$ is simply a Newtonian attraction and γ/x^3 a repulsion from the origin. Far from the origin the first term predominates and so we have essentially attraction, while near the origin the second term predominates and we have repulsion. The equation of motion is

$$m\ddot{x} = \mp \frac{\alpha}{x^2} + \frac{\gamma}{x^3}.$$

With $y = \dot{x}$, the paths are given by

$$\frac{my^2}{2} \mp \frac{\alpha}{x} + \frac{\gamma}{2x^2} = C.$$

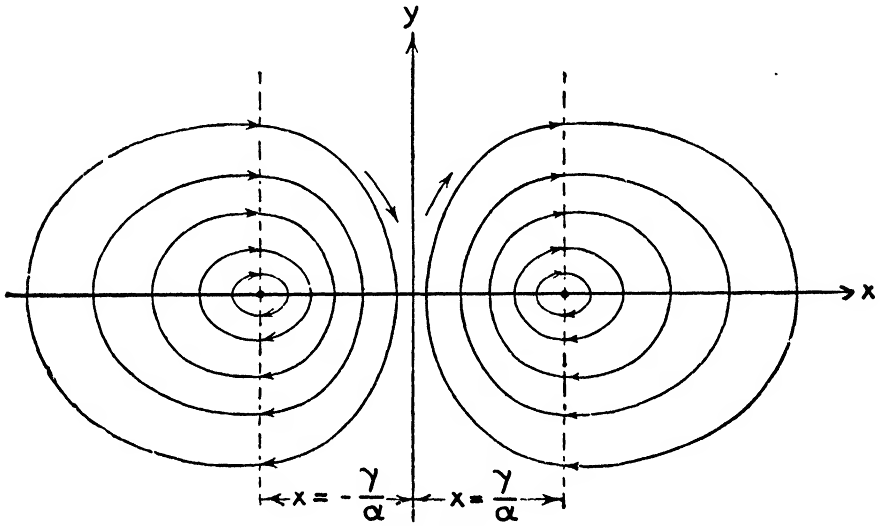


FIG. 68.

The branch point of the parameter α is $\alpha = 0$, and that of the parameter γ is $\gamma = 0$. There are two singular points $x = \gamma/\alpha$ and $x = -\gamma/\alpha$, which are always stable when α and γ are positive. The form of the paths for both α and $\gamma \neq 0$ is represented in Fig. 68, for $\alpha = 0$ in Fig. 69, and for $\gamma = 0$ in Fig. 70. In the first case, the point can oscillate around one of the equilibrium states without crossing the origin. When $\gamma = 0$, the point is attracted everywhere. It is necessary to remember, however, that the law of interaction is not valid at the origin. Therefore our discussion can only be applied at the points other than the origin.

4. Motion of a conductor carrying a current. Consider an unlimited rectilinear conductor through which there passes a current I. It is supposed to attract a conductor AB of length l through which

there passes a current i. The wire AB has a mass m and is attracted to the equilibrium position by the spring force kx, proportional to the distance of the wire from the equilibrium state O situated at the distance a from the first wire (Fig. 71). Let us assume that the current at the extremities of the wire AB is coming in and out through

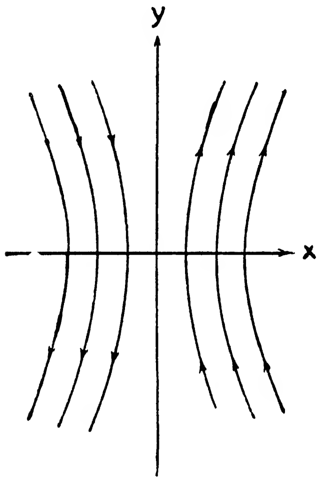


FIG. 69.

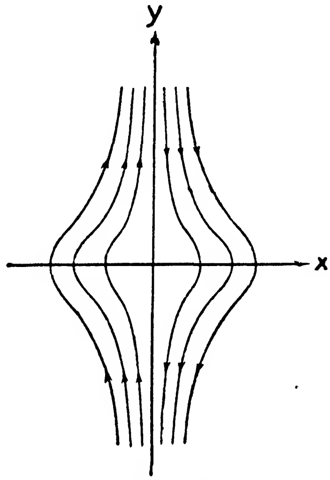


FIG. 70.

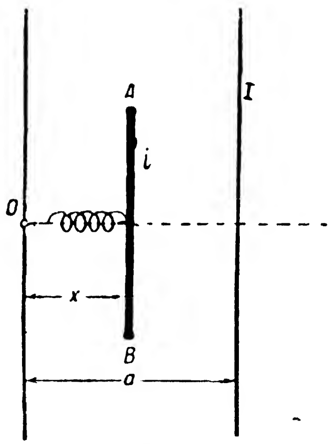


FIG. 71.

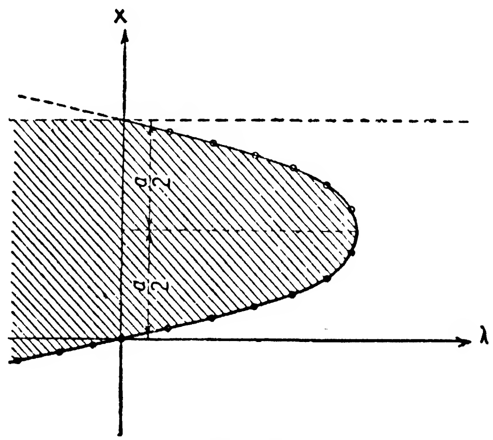


FIG. 72.

leads perpendicular to the current I, so that the interaction force between the wires can be written

$$f_1 = \frac{2Iil}{d}$$

where d is the distance between the wires. Then the force acting on AB can be written

$$-\phi(x,\lambda) = \frac{\partial V}{\partial x} = kx - \frac{2Iil}{a - x} = k\left(x - \frac{\lambda}{a - x}\right),$$

where $\lambda = 2Iil/k$. The position of equilibrium is therefore given by

$$-\phi(x,\lambda) = k\left(x - \frac{\lambda}{a - x}\right) = 0 \qquad \text{or} \qquad x^2 - ax + \lambda = 0.$$

The curve $\phi(x,\lambda) = 0$ is represented in Fig. 72. The equation

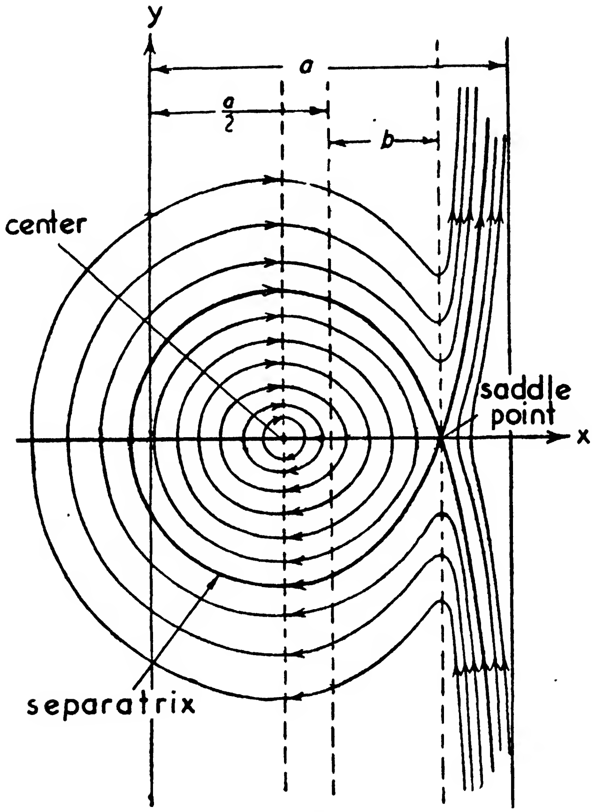


Fig. 73.

$\phi(x,\lambda) = 0$ in x has equal roots for $\lambda = a^2/4$. Hence for $x = a/2$ and $\lambda = a^2/4$, both $\phi(x,\lambda)$ and $\phi_x(x,\lambda)$ vanish, so that $\lambda = a^2/4$ is a branch point. The equations of motion are

$$\frac{dx}{dt} = y, \qquad \frac{dy}{dt} = -\frac{k}{m}\left(x - \frac{\lambda}{a - x}\right) = \frac{k}{m}\left(\frac{x^2 - ax + \lambda}{a - x}\right),$$

and hence

$$\frac{dy}{dx} = \frac{k}{m}\,\frac{x^2 - ax + \lambda}{(a - x)y}.$$

We have here not only a singular point but also a "singular line" $x = a$. It should be kept in mind that, when $x > a$, f_1 is represented by the term $2Iil/(a - x)$ and not by $2Iil/(x - a)$. However, the case $x > a$ has no physical interest. The energy integral is

$$\frac{my^2}{2} + \frac{1}{2}kx^2 + k\lambda \log (a - x) = C.$$

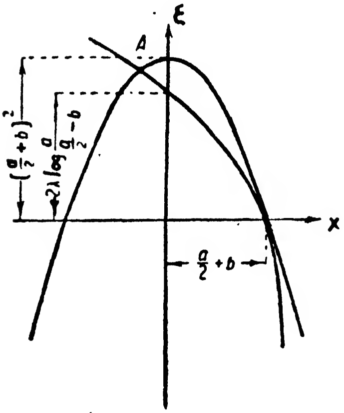


FIG. 74.

Consider first the case $\lambda < a^2/4$ (Fig. 73). There are two singular points, both on $y = 0$: a center $x = (a/2) - b$, $b = \sqrt{(a^2/4) - \lambda}$, and a saddle point $x = (a/2) + b$. The tangents to the paths are vertical on the x-axis and also on the line $x = a$, and horizontal on the vertical lines passing through the two singular points. The singular line $x = a$ is the common asymptote of all the other infinite paths. The separatrix is obtained by setting in the energy integral $y = 0$ and $x = (a/2) + b$ (i.e. the conditions that the separatrix pass through the saddle point). This yields for C the value

$$C_0 = \frac{k}{2}\left(\frac{a}{2} + b\right)^2 + k\lambda \log \left(\frac{a}{2} - b\right).$$

Hence the equation of the separatrix is

$$\frac{my^2}{2} + \frac{k}{2}\left(x^2 - \left(\frac{a}{2} + b\right)^2\right) + k\lambda \log \frac{a - x}{\frac{a}{2} - b} = 0.$$

The second root x of this equation for $y = 0$ (i.e. the coordinate of the intersection point of the separatrix and the x-axis) can be found by using the graphical construction indicated in Fig. 74. From the two graphs

$$\xi = \left(\frac{a}{2} + b\right)^2 - x^2, \qquad \xi = 2\lambda \log \frac{a - x}{\frac{a}{2} - b}$$

or $x = a - \left(\frac{a}{2} - b\right) e^{\frac{\xi}{2\lambda}}$, we obtain the second intersection point of these curves other than $x = a/2 + b$, $y = 0$. An examination of the paths (Fig. 73) shows that the segment of conducting wire AB will

oscillate if the initial conditions are such that the representative point is situated at the initial time inside the loop of the separatrix. In particular, when the initial velocity is zero, the segment AB will oscillate if its deviation from the equilibrium position is not too large.

Consider now the case $\lambda > a^2/4$. This time $\phi(x,\lambda) = 0$ has no real roots, and there are no singular points. The paths are represented in Fig. 75. Whatever the initial conditions, the conducting wire AB

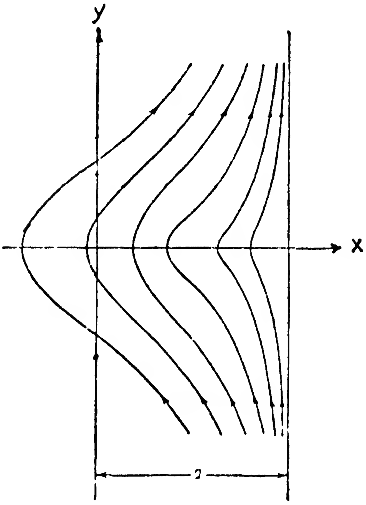


Fig. 75.

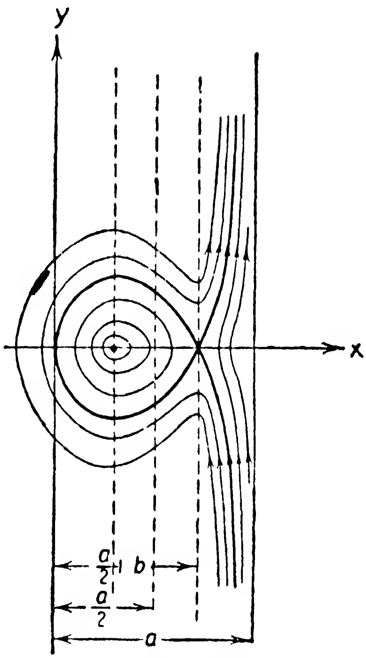


Fig. 76.

finally approaches the line $x = a$ with an indefinitely increasing velocity. There is no oscillating motion.

There is left the intermediary case where $\lambda = a^2/4$. It is easy to see that in the first case, when λ increases, the two singular points approach each other and that for $\lambda = a^2/4$ they coincide, giving rise to a cusp (Fig. 78). Here again periodic motions are impossible. For all initial conditions the conducting wire moves with indefinitely increasing velocity toward the infinite conductor. The branches I and II, passing through the singular point, separate two types of motions. In the motions of the first type, at the initial moment the system is situated in the region bounded by the straight line $x = a$ and the branches I and II, and the conducting wire AB moves toward the straight line $x = a$ without crossing the equilibrium position. In motions of the

second type, at the initial moment the system is outside the region limited by the branches I and II and the straight line $x = a$, and the conducting wire AB always crosses the equilibrium position.

Let us finally examine the case when $\lambda < 0$. The change of sign can be obtained by changing the direction of one of the currents i or I. Here $\phi(x,\lambda) = 0$ always has two real roots $x_{1,2} = (a/2) \pm \sqrt{(a^2/4) - \lambda}$. One of these roots is always negative and the other

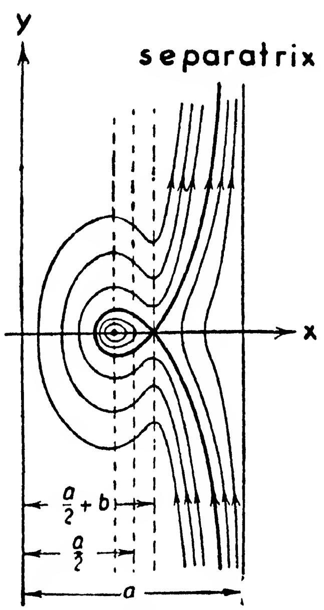


FIG. 77.

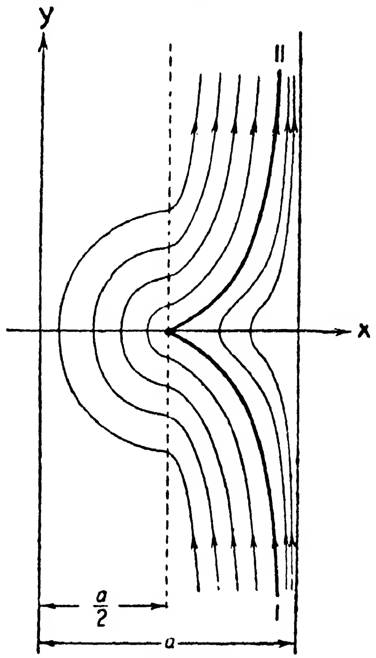


FIG. 78.

$> a$, and hence of no interest. The negative root yields a center and oscillations may occur.

§6. EQUATIONS OF MOTION

When dealing with complicated conservative systems it is often best to set up the equation of motion by means of the Lagrangian function. If q is the basic coordinate in a system with one degree of freedom (no others are considered here), then the function is of the form $L(q,\dot{q})$ and the resulting equation of motion is

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q} = 0. \tag{11}$$

As we shall see later, Lagrange's equations are particularly convenient for deriving the equations of motion of electromechanical systems.

It may be observed, as one may verify directly, that here the energy integral assumes the form

$$\dot{q}\frac{\partial L}{\partial \dot{q}} - L = h.$$

1. Hamilton's equation. Side by side with q introduce the momentum $p = \partial L/\partial \dot{q}$ and the Hamiltonian function

$$H = p\dot{q} - L = H(p,q). \tag{12}$$

In terms of H one may reduce the equation of motion (11) to two differential equations of the first order:

$$\dot{q} = \frac{\partial H}{\partial p}; \qquad \dot{p} = -\frac{\partial H}{\partial q} \tag{13}$$

known as Hamilton's equations. The Hamiltonian form of the equations of motion offers many advantages in dynamical problems.

For Hamilton's equation the "energy integral" may be written

$$H(p,q) = h. \tag{14}$$

2. Integral invariants. Let our dynamical system be determined by

$$\dot{x} = P(x,y), \qquad \dot{y} = Q(x,y). \tag{15}$$

Let the paths be considered as lines of flow (so-called laminar flow) in a liquid of a certain density $\rho(x,y)$ (mass per unit area) and velocity given by (15). If $G(t_0)$ is any region at time t_0, the mass in the region is

$$I(t_0) = \iint_{G(t_0)} \rho(x_0,y_0)\, dx_0\, dy_0.$$

At time t the points in $G(t_0)$ will be found in a new region $G(t)$ and their mass will be

$$I(t) = \iint_{G(t)} \rho(x,y)\, dx\, dy. \tag{16}$$

We say that the equation of motion admits an integral invariant (16) if the density $\rho(x,y)$ can be so chosen that the mass of liquid remains constant during the motion, independently of the initial $G(t_0)$, i.e. if

$$\frac{d}{dt}\iint_{G(t)} \rho(x,y)\, dx\, dy = 0.$$

This relation expresses the condition of invariance of mass under the flow. It is shown in hydrodynamics that it is equivalent to the so-called equation of continuity:

$$\frac{\partial(\rho P)}{\partial x} + \frac{\partial(\rho Q)}{\partial y} = 0. \tag{17}$$

In particular, the Hamiltonian system (13) has the ordinary area ($\rho = 1$)

$$I(t) = \iint dp\,dq$$

for integral invariant, since (17) reduces to the identity

$$\frac{\partial}{\partial q}\frac{\partial H}{\partial p} + \frac{\partial}{\partial p}\left(-\frac{\partial H}{\partial q}\right) = 0.$$

If we return to the variables $q,\dot{q}$, we have

$$\iint dp\,dq = \iint D\,d\dot{q}\,dq = \iint \frac{\partial^2 L}{\partial \dot{q}^2}\,d\dot{q}\,dq$$

where

$$D = -\begin{vmatrix} \dfrac{\partial p}{\partial q} & \dfrac{\partial p}{\partial \dot{q}} \\ \dfrac{\partial q}{\partial q} & \dfrac{\partial q}{\partial \dot{q}} \end{vmatrix} = \frac{\partial^2 L}{\partial \dot{q}^2}.$$

Hence Lagrange's equations admit the integral invariant

$$\iint \frac{\partial^2 L}{\partial \dot{q}^2}\,dq\,d\dot{q}$$

wherever $\partial^2 L/\partial \dot{q}^2$ is finite and positive. These conditions must be fulfilled since ρ must be finite and positive.

Example I: Harmonic motion.

$$\dot{q} = p, \qquad \dot{p} = -q.$$

Introduce polar coordinates r,ϕ so that $q = r\cos\phi$, $p = r\sin\phi$. As a consequence

$$r\dot{r} = 0, \qquad r = \text{const.}, \qquad \text{and} \qquad \dot{\phi} = -1, \qquad \phi = -t + \tau.$$

Hence the motion in the p,q-plane is a pure rotation and preserves areas (Fig. 79).

Example II: Motion under a constant force.

$$\begin{aligned} \dot{q} &= p, \qquad & q &= q_0 + p_0 t - \tfrac{1}{2}gt^2; \\ \dot{p} &= -g, \qquad & p &= p_0 - gt. \end{aligned}$$

Take at time $t = 0$ on the phase plane a square with vertices q_0,p_0; $(q_0 + a),p_0$; $q_0,(p_0 + a)$; $(q_0 + a),(p_0 + a)$. The square will buckle more and more with time (Fig. 80); its area will, however, remain constant since the sides parallel to the q-axis, i.e. connecting the points of equal initial velocity p_0, will move parallel to themselves and their length will remain constant and equal to a. Instead of a square with

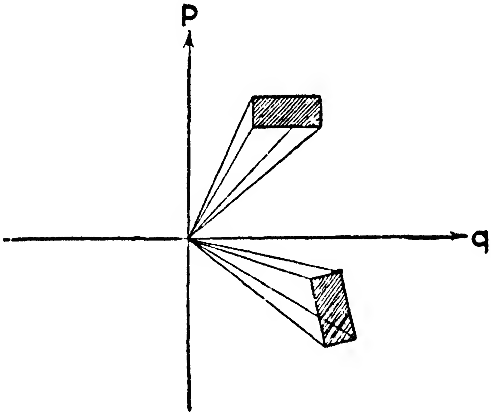


FIG. 79.

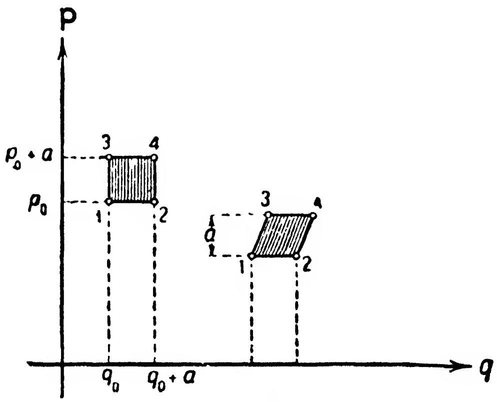


FIG. 80.

sides equal to a we obtain a parallelogram with a base a and altitude a, i.e. with the same area as the square.

3. Oscillating circuit with iron core. As the first example of a non-linear conservative system, we shall examine an electrical oscillatory circuit with a coil containing an iron core (Fig. 81). We neglect the resistance of the circuits and the hysteresis loss and thus may consider the system as conservative. If we disregard the dispersion in the coil, i.e. if we assume that the whole magnetic flux Φ passes

through all the w turns of the coil, we have by Kirchhoff's law

$$\frac{1}{C}\int i\,dt + w\frac{d\Phi}{dt} = 0. \tag{18}$$

Here Φ is a function of i whose graph has the general aspect of Fig. 82, at least if it is assumed, as one often does for simplification, that there is no residual magnetism. There are various approximations to the

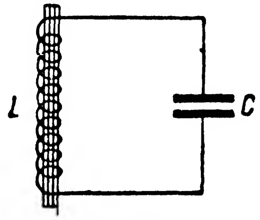


FIG. 81.

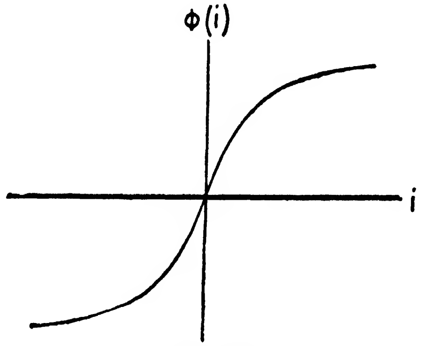


FIG. 82.

graph. A satisfactory one, due to Dreyfus and to be used here, is

$$\Phi(i) = A \arctan\frac{wi}{S} + B\frac{wi}{S} \tag{19}$$

where A,B,S are positive constants.

One may easily put (18) in the Lagrange form. To do so, introduce the charge q of the condenser, so that $i = \dot{q}$, and set

$$L = w\int \Phi(\dot{q})\,d\dot{q} - \frac{q^2}{2C}.$$

Thus we obtain

$$\frac{\partial L}{\partial \dot{q}} = w\Phi(\dot{q}), \qquad \frac{\partial L}{\partial q} = -\frac{q}{C}$$

and (18) does take the Lagrangian form

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q} = 0.$$

The "energy integral" is

$$\dot{q}\frac{\partial L}{\partial \dot{q}} - L = h,$$

or here

$$w\Phi(\dot{q})\dot{q} - w\int \Phi(\dot{q})\,d\dot{q} + \frac{q^2}{2C} = h \tag{20}$$

where h actually represents the total energy of the system. In fact, the electrostatic energy in the condenser is $V = q^2/2C$. The magnetic energy T in the coil is defined as the work against the e.m.f. of self-induction

$$T = w\int \frac{d\Phi(i)}{dt}\,\dot{q}\,dt = w\int \dot{q}\,d\Phi(\dot{q}). \tag{21}$$

By partial integration we find

$$T = w\Phi(\dot{q})\dot{q} - w\int \Phi(\dot{q})\,d\dot{q}. \tag{22}$$

Consequently $h = T + V$. Since here $L \neq T - V$, we have an example of a Lagrange function not equal to the difference between the kinetic and the potential energy. If we introduce the new variable $p = \partial L/\partial\dot{q} = w\Phi(\dot{q})$, we can reduce (20) to the Hamiltonian form. The Hamilton function is

$$H(p,q) = \int \Psi(p)\,dp + \frac{q^2}{2C}$$

where $\dot{q} = \Psi(p)$ is the inverse function obtained from $p = w\Phi(\dot{q})$. The graph of Fig. 82 shows that both Φ and Ψ are monotonic increasing. Hamilton's system is here

$$\dot{p} = -\frac{\partial H}{\partial q} = -\frac{q}{C}; \qquad \dot{q} = \frac{\partial H}{\partial p} = \Psi(p).$$

The integral invariant is

$$\iint dp\,dq = \iint \frac{\partial^2 L}{\partial \dot{q}^2}\cdot d\dot{q}\,dq = \iint \frac{d\Phi(\dot{q})}{d\dot{q}}\,d\dot{q}\,dq,$$

and consequently the role of the density is played by the quantity $\frac{d\Phi(\dot{q})}{d\dot{q}}$ which is the slope of the graph of Fig. 82 and hence always positive. Referring to (20), (21) and (22), the energy integral may be written

$$w\int \frac{d\Phi}{d\dot{q}}\,\dot{q}\,d\dot{q} + \frac{q^2}{2C} = \text{constant}.$$

This expression is analogous to those obtained for the conservative system in §5 with the sole difference that q and $\dot{q}$ are interchanged. Therefore, concerning the integral curves, we can make the same statements as for the simpler conservative systems. The expression under the integral sign is always positive and its derivative vanishes only at the point $\dot{q} = 0$. Consequently $q = 0$ corresponds to the minimum of the energy and the singular point $q = 0$, $\dot{q} = 0$ is a center and it

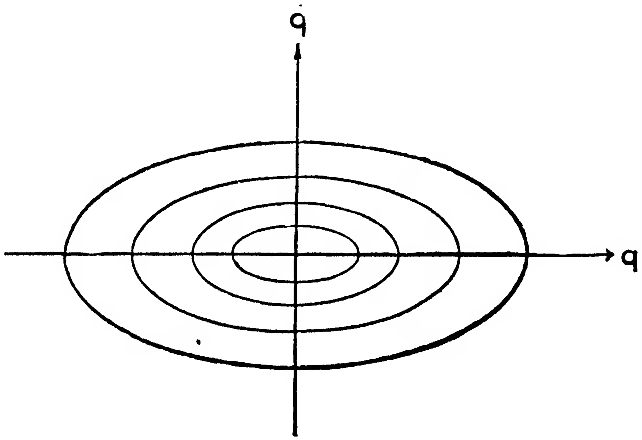


FIG. 83.

corresponds to stable equilibrium. All the paths are concentric ovals around the singular point. From (19) follows

$$\frac{d\Phi}{d\dot{q}} = \frac{d\Phi}{di} = \frac{Aw}{S}\,\frac{1}{1 + \dfrac{w^2\dot{q}^2}{S^2}} + B\,\frac{w}{S},$$

$$w\int \frac{d\Phi}{d\dot{q}}\,\dot{q}\,d\dot{q} = \frac{Aw^2}{S}\int \frac{\dot{q}\,d\dot{q}}{1 + \dfrac{w^2\dot{q}^2}{S^2}} + B\,\frac{w^2}{S}\int \dot{q}\,d\dot{q}.$$

By obvious integration we obtain then the equation of the paths in finite form:

$$\frac{AS}{2}\log\left(\frac{\dot{q}^2}{2} + \frac{S^2}{2w^2}\right) + \frac{Bw^2}{2S}\,\dot{q}^2 + \frac{q^2}{2C} = \text{constant}.$$

Fig. 83 represents a family of these curves constructed for particular values of the parameters.

4. Oscillatory circuit with capacitance depending on the charge. When the dielectric consists of Rochelle salt, the capacitance $C(q)$

depends on the charge q after a manner shown in Fig. 84. A reasonable expression for the capacitance is

$$C(q) = \frac{C_0}{1 + Aq + Bq^2}. \tag{23}$$

The lack of symmetry is caused in substance by a "residual" charge analogous to residual magnetism. Be it as it may, neglecting ohmic resistances and hysteresis losses, since C is a function of q alone we have a non-linear conservative system. By Kirchhoff's law

$$L_0\ddot{q} + \frac{q}{C(q)} = 0. \tag{24}$$

To reduce this relation to the Lagrange type, set

$$L = \frac{L_0\dot{q}^2}{2} - \int \frac{q\,dq}{C(q)}.$$

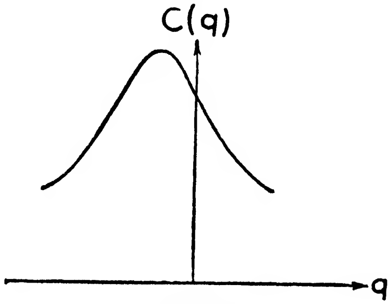


FIG. 84.

Here then

$$\frac{\partial L}{\partial \dot{q}} = L_0\dot{q}, \qquad \frac{\partial L}{\partial q} = -\frac{q}{C},$$

which reduces (24) to the desired form

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q} = 0.$$

The energy integral is then

$$\frac{L_0\dot{q}^2}{2} + \int \frac{q}{C(q)}\,dq = h. \tag{25}$$

Since the energy of the charge of the condenser is equal to the work of the current charging the condenser,

$$V = \int \frac{q}{C(q)}\,\dot{q}\,dt = \int \frac{q\,dq}{C(q)},$$

and hence h represents the total energy. This case differs from the previous one in that the Lagrange function $L = T - V$, i.e. it is equal to the difference between the magnetic and electromagnetic

energies of the system. The reduction to a Hamiltonian system offers no difficulty. The relation (25) represents a family of paths in the phase plane $q,\dot{q}$. Since the function $\int (q/C(q))\,dq$ is minimum when $q = 0$, the point $q = 0$, $\dot{q} = 0$ is a center. By means of (23) we reduce (25) to

$$\frac{L_0\dot{q}^2}{2} + \frac{q^2}{2C_0} + \frac{Aq^3}{3C_0} + \frac{Bq^4}{4C_0} = h.$$

This equation defines a family of concentric ovals (Fig. 85) which are non-symmetric with respect to the $\dot{q}$-axis owing to the term in q^3.

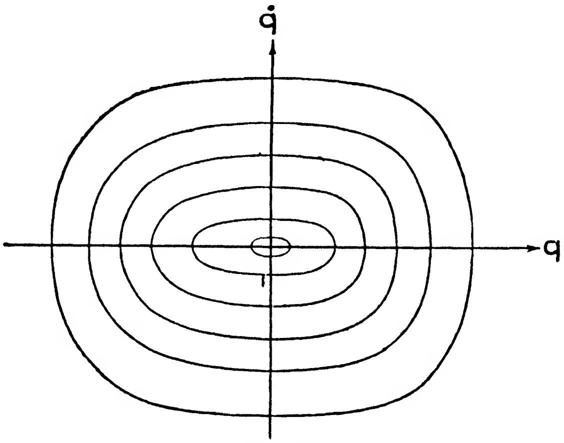


FIG. 85.

§7. PERIODIC MOTIONS IN CONSERVATIVE SYSTEMS

The periodic motions are next in importance to the positions of equilibrium. Their most noteworthy property is that in conservative systems with one degree of freedom they are never isolated and always form part of a continuous family of such motions. We propose to discuss the stability of the periodic motions.

To a periodic motion there corresponds a closed path γ in the phase plane. Take a point P_0 at time t_0 on γ. Then γ will be stable in the sense of Liapounoff whenever corresponding to any $\epsilon > 0$ there is a $\delta(\epsilon) > 0$ such that, if a point Q_0 is nearer than $\delta(\epsilon)$ to P_0 at time t_0, then the points P,Q into which P_0,Q_0 will go at time t will be nearer to one another than ϵ whatever $t > t_0$. A somewhat less precise formulation is this: Let the representative point be white and imagine that it becomes black (Fig. 86) under the effect of the disturbance. Then the initial undisturbed motion, i.e. the motion which would have taken place if there were no disturbance, will be represented by the white

point and the disturbed motion by the black point. Now the condition of stability can be formulated in the following way. If at time t_0 the black point is sufficiently close to the white point (i.e. the disturbance is sufficiently small), it remains quite close to it ever after.

It is easy to see that a periodic motion in a conservative system is generally unstable, for in general the period of revolution of the representative point around the various closed paths is different. Consequently the white and the black points will diverge more and more, however close they may be at the outset. It is true that later they will again approach one another, but for any small initial distance different from zero we cannot in general "retain" them at a small distance apart. The distance between the white and black points will fail to increase with respect to the initial distance only in the special case when the white and black points move along the same path, i.e. when the disturbance is such that the representative point under the effect of the disturbance is moved along the same path.

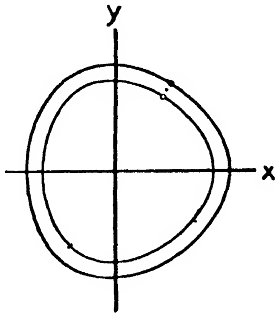


Fig. 86.

Periodic motions in conservative systems will be stable according to Liapounoff only in the special case when isochronism takes place, i.e. when the period of revolution is the same for different paths. But even in this case we do not have absolute stability, i.e. paths of the representative points approaching asymptotically after a sufficiently small disturbance. Such paths do not occur generally in conservative systems with one degree of freedom, and we shall meet them only in non-conservative systems. Although periodic motions in conservative systems are then generally unstable in the sense of Liapounoff, they do possess a certain form of stability, namely, a sufficiently close trajectory is always located entirely in the immediate vicinity of the given trajectory. This type of stability is called *orbital.* It plays an important role in the general theory of the behavior of the paths.

1. Some general considerations regarding conservative systems. Broadly speaking, the paths in a conservative system have given rise to a family of curves in the phase plane represented by an equation

$$F(u,v) = C \tag{26}$$

where F has a fixed sign. We may assume for convenience that $F \geqslant 0$, thus allowing for C only non-negative values. In our systems F was the Hamiltonian function and (26) the energy integral.

Let us take as our starting point a family of curves (26) without reference to any dynamical system. To simplify matters we assume F analytical for all u,v and again never negative. As for u,v, they may be parametric coordinates on a surface (usually a plane or cylinder). We shall refer to the value of C as a "level" and the curves (26) are thus the curves of constant level. In particular, if u,v are plane coordinates, we may think of the plane as horizontal and of C as a vertical coordinate (the same as the usual z in a three-space coordinate system x,y,z). The three-space is thus referred to the coordinates

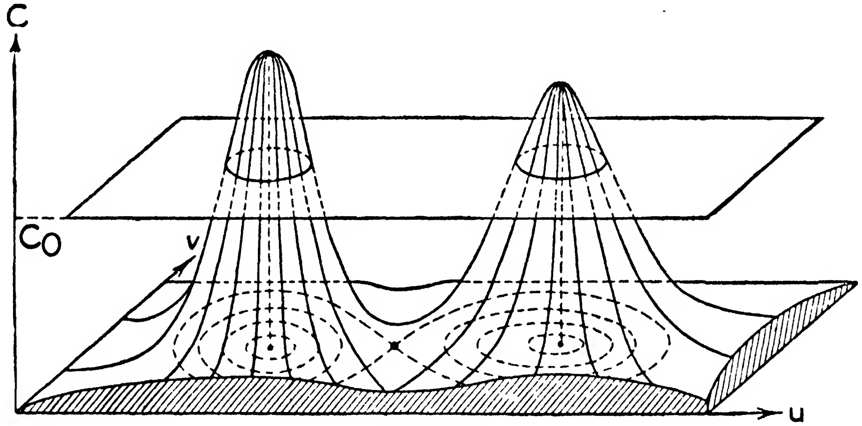


FIG. 87.

x,y,C. Then (26) represents a surface in that space whose horizontal sections are the curves of our family.

When u,v are coordinates on a right circular cylinder, generally one of them, say u, is an angle of rotation about the axis and the other a displacement parallel to the axis.

Let $\Phi(z)$ be an analytical function which is monotonic and whose sign is the same as that of z. Then $\Phi(F(u,v)) = C'$ represents the same family as (26). For to specify a numerical value of C' is to specify one value C of z in $\Phi(z) = C'$, hence one curve $F(u,v) = C$. To replace F by Φ amounts essentially to a change of vertical C scale.

The differential equation of the curves (26) is

$$\frac{dv}{du} = -\frac{F_u}{F_v}. \tag{27}$$

The singular points correspond to the values of u,v for which $F_u = F_v = 0$. It can happen that F_u,F_v vanish simultaneously not only in isolated points but also along a certain analytical curve. Let us show that such a curve is necessarily an integral curve, i.e. that its points

satisfy the equation $F(u,v) = \text{constant}$. Assume the curve given in parametric form by $u = u(s)$, $v = v(s)$. Then

$$\frac{dF}{ds} = F_u \cdot \frac{du}{ds} + F_v \cdot \frac{dv}{ds} = 0$$

since the partials at the left vanish. Hence $F = \text{constant}$, i.e. $F(u,v)$ is constant along the curve. It is easy to see that this case can take place if the curve consists of points where the tangent plane is parallel to the phase plane, for example, when the surface $F(u,v) = C$ has the form of a crater whose edges are situated on the same level (Fig. 88). There cannot occur isolated singular points through which there pass infinitely many curves of the family. For the curves would fill a

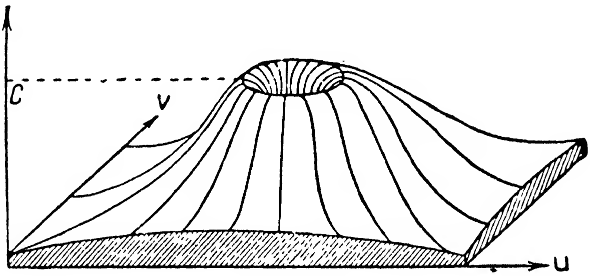


FIG. 88.

portion of the plane and yet be of the same level, hence $F(u,v)$ would be constant. Thus nodes and foci are ruled out. Similarly none of the curves is an oval around which other curves spiral. Finally one may affirm that, if there exists one oval, there must exist a whole continuous family of them entirely filling a portion of the plane.

Let us endeavor now to consider the family (26) as a family of paths in a dynamical system. Such a system, if it exists, will still be referred to henceforth also as "conservative." This is justified on the ground of the similarity of the properties of the "paths" with those of conservative systems in the earlier sense. Of course, we continue to deal exclusively with systems of one degree of freedom. All this would be meaningless in systems with more degrees of freedom.

It is presumed then that (27) results from the elimination of dt from the equations of a dynamical system. To restore dt we write naturally

$$\frac{du}{F_v} = -\frac{dv}{F_u} = \frac{dt}{Q(u,v)}$$

and $S = 1/Q$ is a certain function "scaling" the time. The equations

of motion in the general form can be written

$$(28)\qquad \begin{cases} \dfrac{du}{dt} = \dfrac{\partial F}{\partial v} \cdot \dfrac{1}{Q(u,v)} = U(u,v), \\ \dfrac{dv}{dt} = -\dfrac{\partial F}{\partial u} \cdot \dfrac{1}{Q(u,v)} = V(u,v). \end{cases}$$

We shall assume that $S(u,v)$ is analytical over the whole u,v plane and has a fixed sign, being never zero, and also that the phase plane either represents an Euclidean plane or else is applicable on an Euclidean plane (a cylinder, for example), and that u and v are rectangular coordinates.

From (28) follows $F_v = QU$, $F_u = -QV$ and hence

$$\frac{\partial(QU)}{\partial u} + \frac{\partial(QV)}{\partial v} = 0.$$

This is the equation of continuity for a steady flow with velocity given by (28) and density Q. Hence

$$\iint Q(u,v)\, du\, dv$$

is an integral invariant. It is easily shown that

$$\iint Q(u,v)\Phi(F(u,v))\, du\, dv$$

where $\Phi(F)$ is analytical in F and of fixed sign, is likewise an integral invariant, and that there are no others.

Conversely, if there exists an analytical function of constant sign $Q(u,v)$ such that (28) holds, then as is well known there is a function F related to it such as we have described and (27) represents a steady flow, or a dynamical system which we still consider as conservative. Sometimes we shall refer to such a system as "generalized" conservative. The relation $F = C$ is the "energy integral" of the system.

Remarks. I. We have seen that when we have an actual phase plane (not some other surface), the closed paths make up sets of concentric ovals each surrounding a singular point. It is important to bear in mind that this need not hold when the phase surface is, say, a cylinder. Thus the horizontal sections of a cylinder do form a family of ovals which do not surround a singular point.

II. The slightest change in the differential equations may completely alter the system and in particular make it cease to be conservative. Thus when $h = 0$,

$$\ddot{x} + h\dot{x} + \omega^2 x = 0$$

defines a harmonic oscillator, which is a conservative system. How-

ever, for h arbitrarily small but positive, the system is dissipative and hence non-conservative. For $h = 0$ the origin is a center but for $h > 0$ and very small it is a focus, i.e. of completely different nature.

III. *Example of a generalized conservative system.* In mechanical or electrical systems the question of conservativeness is readily settled: when there is no friction or resistance the system is conservative. We shall discuss an example drawn from biology in which conservativeness, in the general sense, can only be decided by reference to an integral invariant.

This example due to Volterra concerns the coexistence of two types of animals, for example two types of fishes. The first type feeds upon the products of the medium which we assume are always present in sufficient quantity. The second type feeds exclusively upon fishes of the first type. The numbers of each type are of course integers and can vary only by jumps, but to apply our general methods we shall consider them as continuous functions of time. Designate by N_1, N_2 the numbers of animals of the first and second types. We assume that, if the first type existed alone, the number of animals would continually grow with a velocity proportional to their number. Thus

$$\dot{N}_1 = \epsilon_1 N_1, \qquad \epsilon_1 > 0.$$

The growth coefficient ϵ_1 depends upon the mortality and the birth rates. If the second species existed alone, it would progressively starve out. For this type a natural law is

$$\dot{N}_2 = -\epsilon_2 N_2, \qquad \epsilon_2 > 0.$$

Assume now that both species live together. Then clearly ϵ_1 will become smaller as N_2 becomes larger. We shall make the simplest possible assumption, namely, that ϵ_1 decreases proportionally to N_2. Similarly we may assume that ϵ_2 varies proportionally to N_1. As a consequence we may write

$$\dot{N}_1 = N_1(\epsilon_1 - \gamma_1 N_2); \qquad \dot{N}_2 = -N_2(\epsilon_2 - \gamma_2 N_1)$$

where ϵ_1, ϵ_2, γ_1, and γ_2 are all positive constants.

Multiplying the first equation by γ_2 and the second by γ_1 and adding, we obtain

$$\gamma_2 \dot{N}_1 + \gamma_1 \dot{N}_2 = \epsilon_1 \gamma_2 N_1 - \epsilon_2 \gamma_1 N_2.$$

Multiplying the first equation by ϵ_2 / N_1 and the second by ϵ_1 / N_2 and adding, we have

$$\epsilon_2 \frac{1}{N_1} \dot{N}_1 + \epsilon_1 \frac{1}{N_2} \dot{N}_2 = -\epsilon_2 \gamma_1 N_2 + \epsilon_1 \gamma_2 N_1.$$

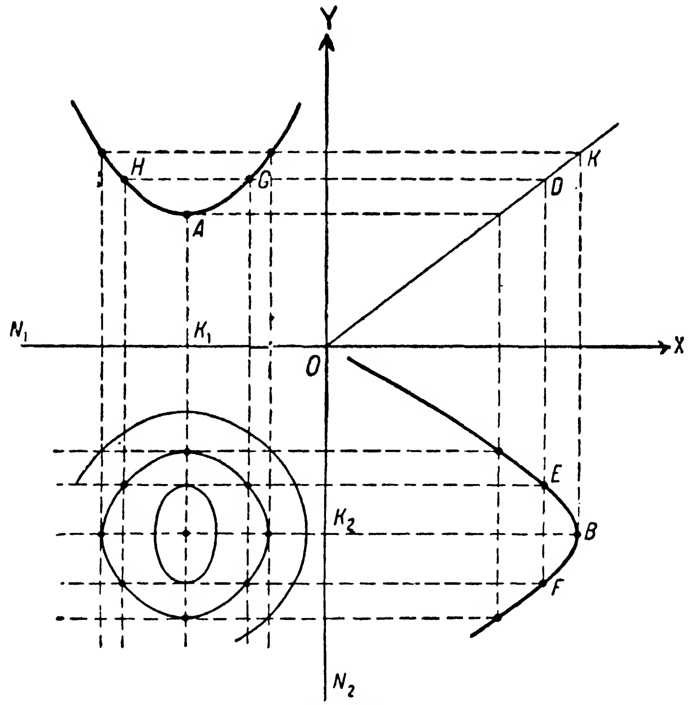


FIG. 89.

Hence

$$\gamma_2\dot{N}_1 + \gamma_1\dot{N}_2 - \epsilon_2 \frac{d \log N_1}{dt} - \epsilon_1 \frac{d \log N_2}{dt} = 0.$$

Integrating this relation we find

$$\gamma_2 N_1 + \gamma_1 N_2 - \epsilon_2 \log N_1 - \epsilon_1 \log N_2 = \text{constant}.$$

This integral can also be written in the form

$$(29) \qquad F(N_1,N_2) = e^{-\gamma_2 N_1} \cdot e^{-\gamma_1 N_2} N_1^{\epsilon_2} N_2^{\epsilon_1} = \text{constant}.$$

It is easy to see that the expression

$$\iint \frac{dN_1\, dN_2}{N_1 N_2}$$

is an integral invariant. Thus we may conclude that our system is conservative. Let us examine now the form of the paths. We shall first reduce (29) to the form

$$N_1^{-\epsilon_2} e^{\gamma_2 N_1} = C N_2^{\epsilon_1} e^{-\gamma_1 N_2}$$

and construct the curves

$$Y = N_1^{-\epsilon_2} e^{\gamma_2 N_1}, \qquad X = N_2^{\epsilon_1} e^{-\gamma_1 N_2}$$

and our path is defined by $Y = CX$. These curves are drawn in Fig. 89 by means of the table

N_1	0	$\frac{\epsilon_2}{\gamma_2}$	$+\infty$	N_2	0	$\frac{\epsilon_1}{\gamma_1}$	$+\infty$
$\frac{dY}{dN_1}$	$-$	0	$+$	$\frac{dX}{dN_2}$	$+$	0	$-$
Y	$+\infty$ $\searrow$	min.	$\swarrow$ $+\infty$	X	0 $\nearrow$	max.	$\nwarrow$ 0

where

$$\frac{dY}{dN_1} = Y\left(-\frac{\epsilon_2}{N_1} + \gamma_2\right), \qquad \frac{dX}{dN_2} = X\left(\frac{\epsilon_1}{N_2} - \gamma_1\right).$$

The construction of the curves, point by point by means of the auxiliary line OK: $Y = CX$, is sufficiently clear from Fig. 89. With

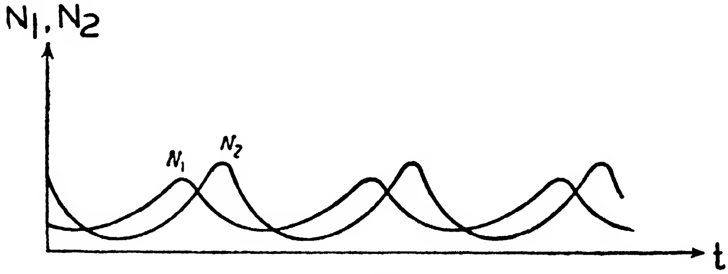


FIG. 90.

the exception of an (irrelevant) curve corresponding to the axes, the integral curves make up a family of concentric ovals with a center at the point $N_1 = \epsilon_2/\gamma_2$, $N_2 = \epsilon_1/\gamma_1$. Hence the law of variation of N_1 and N_2 is periodic (Fig. 90).

CHAPTER III

Non-Conservative Systems

§1. Introduction

We have already considered both non-conservative linear systems and general (i.e. linear or non-linear) conservative systems. The former do not admit periodic motions, but the latter may admit arbitrarily many, their amplitudes being determined by the initial conditions. We are particularly interested in systems where the amplitudes of the periodic motions are determined by the system itself and, moreover, do not vary very much when the system is changed a good deal. Conservative systems are not generally of this type. Later we shall see that only non-conservative non-linear systems are physically significant. In this chapter we shall describe two fundamental types of such systems, called *dissipative* and *self-oscillating*.

§2. Dissipative Systems

In contrast with the conservative case, we have here potential free forces. Hence, after introducing "generalized forces," the Lagrange equation may be written:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q} - \Phi = 0 \tag{1}$$

where Φ, the generalized force, is usually a function of $\dot{q}$.

If the non-conservative forces are like friction, they reduce the energy and hence are of opposite sign from, the velocity. Thus

$$\Phi \dot{q} \leqslant 0 \tag{2}$$

and equality holds only when $\dot{q} = 0$, i.e. only in the equilibrium state. Multiplying then (1) by $\dot{q}$ we find

$$\dot{W} - \Phi \dot{q} = 0. \tag{3}$$

Usually W is the total energy of the system, and by (2) and (3) this is a decreasing function. Hence, if W does not approach $-\infty$, it has a limiting value W_0, and $\Phi \dot{q} \to 0$. Thus for systems satisfying (2) the equilibrium states are the only stationary states. We call such systems *dissipative*. Since the energy always decreases in such systems, they do not admit periodic motions.

As an example of a dissipative system consider the large deviations of a pendulum subject to friction. We shall assume $\Phi = -b\dot{q}$ with $b > 0$. The Lagrangian L has the form:

$$L = \frac{I\dot{\phi}^2}{2} + mgl(\cos\phi - 1)$$

and the Lagrange equation becomes

$$I\ddot{\phi} + b\dot{\phi} + mgl\sin\phi = 0. \tag{4}$$

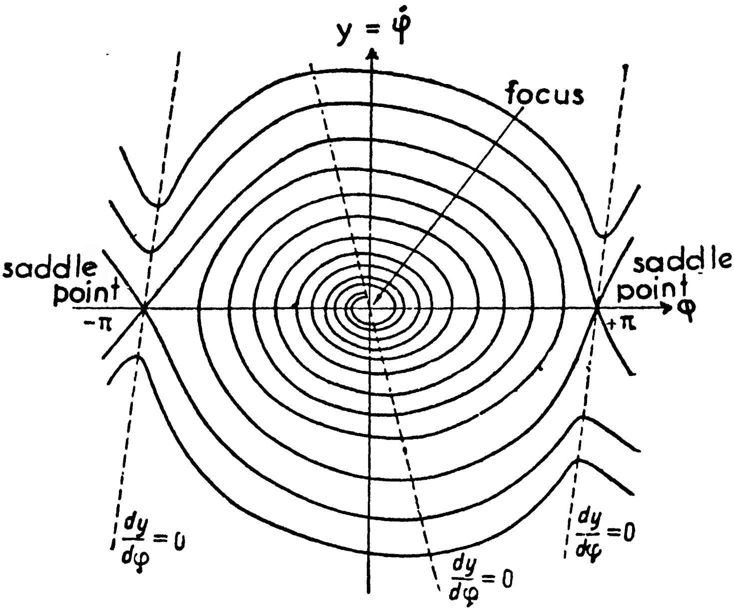


FIG. 91.

or equivalently

$$\dot{\phi} = y, \qquad I\dot{y} = -by - mgl\sin\phi. \tag{4'}$$

The image in the phase plane is defined by

$$\frac{dy}{d\phi} = -\frac{b}{I} - \frac{mgl\sin\phi}{Iy}, \qquad y = \dot{\phi}. \tag{5}$$

The *singularities* are $\phi = k\pi$, $y = 0$ (k any integer), and correspond to stable or unstable equilibrium according as k is even or odd. For k even the singularity is a focus (Fig. 91) if $b^2 < 4Imgl$ and a node (Fig. 92) if $b^2 > 4Imgl$; for k odd the singularities are saddle points. The tangents to the paths are vertical on the ϕ-axis and horizontal on the curves $y = -(mgl/b)\sin\phi$. Figs. 91 and 92 show that there

are no periodic motions and that for almost all initial conditions the system tends to a stable equilibrium.

If (2) does not hold, the system ceases to be dissipative: energy may be increased at the expense of "friction" since $\Phi\dot{q} > 0$ implies $\dot{W} > 0$. We have already discussed such systems, notably Froude's pendulum. One cannot assert now that periodic motions are ruled

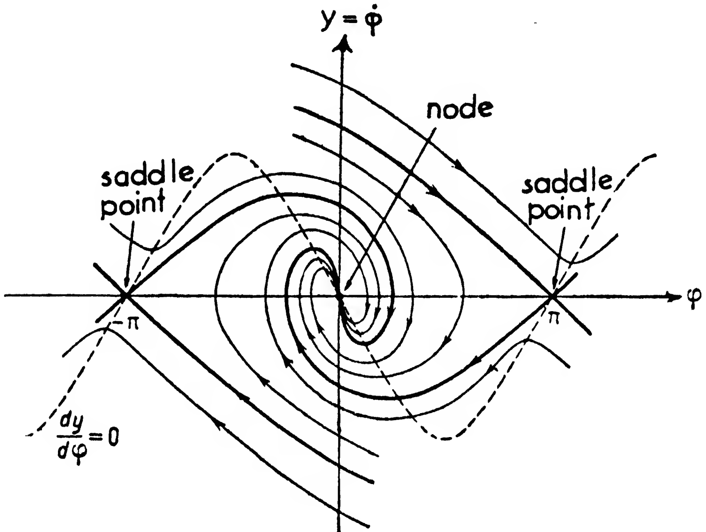


FIG. 92.

out. Thus, if $\Phi = -b\dot{q}^2$, $b > 0$, Φ resists the motion when $\dot{q} > 0$ and assists it when $\dot{q} < 0$. Here

$$\dot{W} + b\dot{q}^3 = 0,$$

and a continuum of periodic motions is possible with amplitudes depending upon the initial conditions.

To elucidate these questions still further, consider the simple equation

$$\ddot{x} + h\dot{x}^2 + x = 0$$

representing an oscillator with dissipation $h\dot{x}^2$. The usual term $\omega_0^2 x$ has been replaced by x $(\omega_0^2 = 1)$ and this can be achieved by changing the time scale (so that t goes into t/ω_0; the old unit is the new unit times ω_0). Our usual passage to a system of two equations of the first order yields here

$$\dot{x} = y, \qquad \dot{y} = -x - hy^2,$$

and hence the equation of the paths is

$$\frac{dy}{dx} = \frac{\dot{y}}{\dot{x}} = -\frac{x + hy^2}{y} \tag{6}$$

or

$$y\frac{dy}{dx} + hy^2 + x = 0.$$

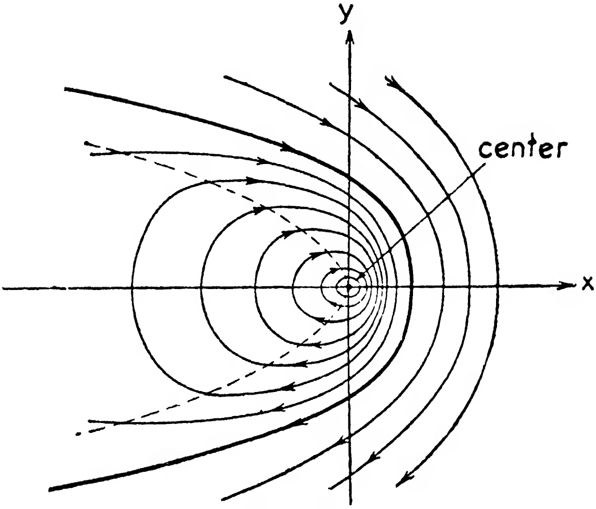


FIG. 93.

The substitution $y^2 = z$ yields

$$\frac{dz}{dx} + 2hz + 2x = 0$$

which is linear with constant coefficients and may be integrated by standard methods. We find

$$z = Ce^{-2hx} - \frac{x}{h} + \frac{1}{2h^2} \tag{7}$$

where C is an arbitrary constant. Hence

$$y^2 = \frac{1}{2h^2}[2Ch^2e^{-2hx} + 1] - \frac{x}{h}. \tag{8}$$

One may readily plot the graph of (7) by adding the ordinates of the exponential curve Ce^{-2hx} and of the line $-(x/h) + (1/2h^2)$. Then the graph of (8) is obtained (Fig. 93) by taking for each x the ordinates $\pm\sqrt{z}$. (See in this connection Chap. II, §3.) The following results are thus obtained:

For $C = -1/2h^2$ the curve reduces to the origin, which is also the only singular point of (6).

For $-1/2h^2 < C < 0$ we have a set of concentric ovals around the origin (Fig. 93).

For $C = 0$ the graph is a separatrix (in the sense that it is an intermediate curve), namely, the parabola

$$y^2 = \frac{1}{2h^2} - \frac{x}{h}.$$

Finally for $C > 0$ the curves have infinite branches.

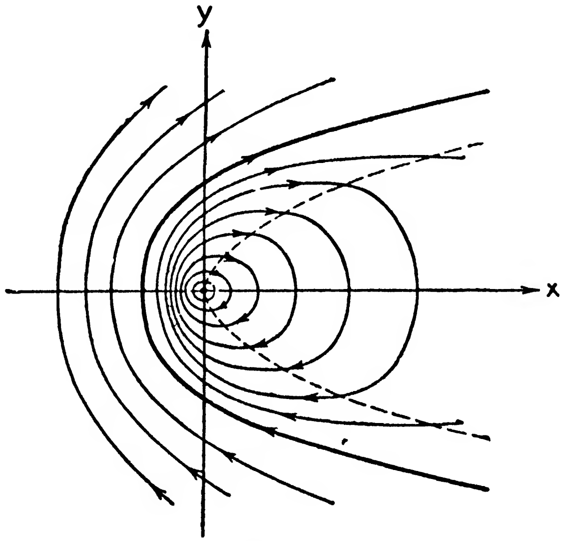


FIG. 94.

If the initial point is interior to the separatrix, the motion is an oscillation (not a harmonic motion) and the behavior of the system is of the type encountered in conservative systems. The singular point is a center.

From the above example one may readily pass to the real situation with true friction proportional to the square of the velocity and resisting the motion. This requires that $h > 0$ for $\dot{x} > 0$ and $h < 0$ for $\dot{x} < 0$. This time the paths for $h < 0$ are merely the reflections of those already obtained in the y-axis (Fig. 94). Hence the paths for true friction ($h\dot{x} = hy > 0$ throughout) are obtained by taking the upper half of Fig. 93 together with the lower half of Fig. 94, with the result shown in Fig. 95. The behavior is exactly like that of a focus in Chapter I, with damped oscillations.

If one has true "negative friction," i.e. $h\dot{x} = hy < 0$ throughout,

then the result is that depicted in Fig. 96, which is the reflection of Fig. 95 in the x-axis. This time the system undergoes expanding oscillations.

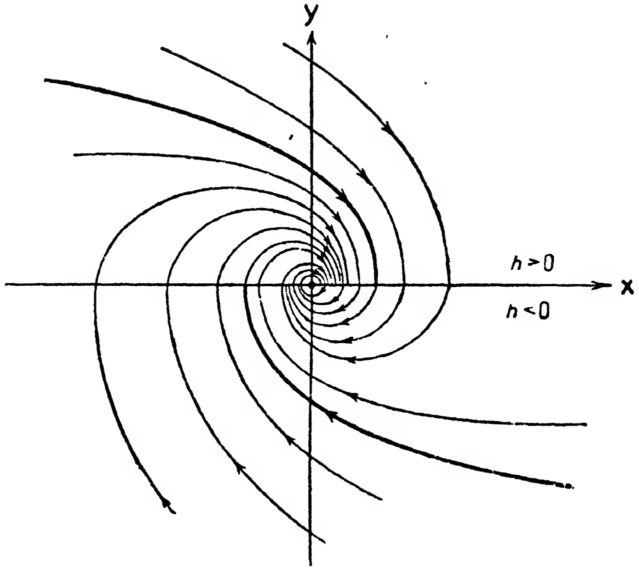


Fig. 95.

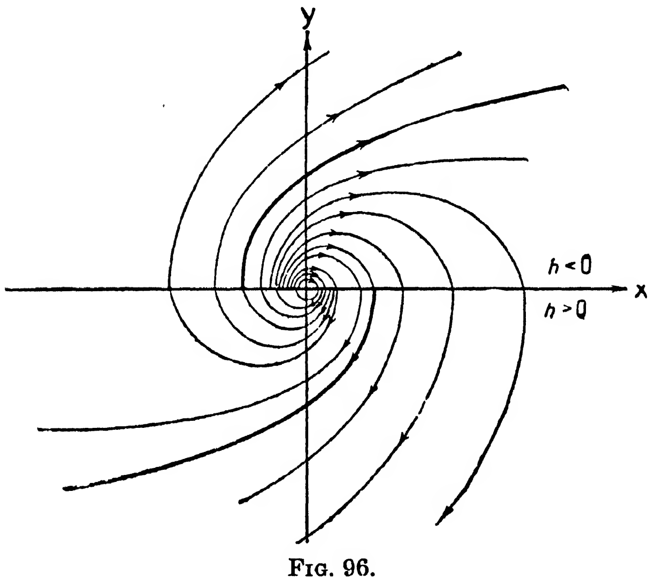


Fig. 96.

§3. Constant Friction (Coulomb Friction)

As another example of a dissipative system we consider the case of "constant friction," i.e. according to Coulomb's law. It is constant in

absolute value but may change sign, since it must always be directed opposite to the velocity. Fig. 97 shows such an example of friction f as a function of the velocity v. If $\pm f_0$ is the constant value of the friction, we admit that, for $v = 0$, f may take any value between

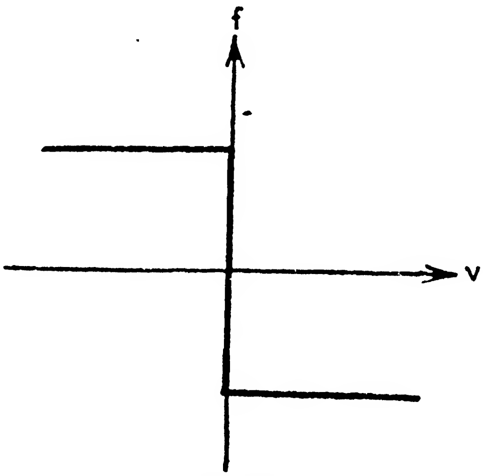


Fig. 97.

$-f_0$ and $+f_0$, the value depending on the other forces acting. Thus consider a cord running over a pulley with a mass m on a horizontal support attached to one end and a mass M suspended at the other end. If the tension Mg is smaller than f_0, the mass m is at rest, so the sum of the forces acting on it—tension plus friction—is zero. Thus the friction may have any value between $-f_0$ and $+f_0$, depending on the tension. If the tension is greater than the friction (in absolute value), then $f_0 = \rho mg$, where ρ is the *coefficient of friction.*[1]

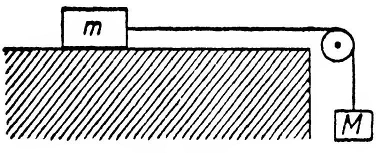


Fig. 98.

The simplest model of an oscillator with Coulomb friction is that shown in Fig. 99. The equation of motion is

$$m\ddot{x} = -kx \pm f_0 \tag{9}$$

where the sign of f_0 is chosen opposite to that of $\dot{x}$. If we let $k = m\omega^2$, $f_0 = am\omega^2$, and let $x_1 = x - a$ when $\dot{x} < 0$, $x_2 = x + a$ when $\dot{x} > 0$,

[1] Sometimes a more general assumption regarding the form of the characteristic of a solid than the Coulomb idealization corresponds better to reality. Thus it is inadequate to assume that the absolute value of the maximum frictional force at rest is greater than the absolute value of the constant frictional force during motion.

we obtain the two equations[1]

$$\ddot{x}_1 + \omega^2 x_1 = 0$$
$$\ddot{x}_2 + \omega^2 x_2 = 0.$$

Hence the motion of the system is obtained by piecing together two halves of harmonic oscillators with equilibrium positions at distances $\pm a$ from the non-frictional equilibrium. Fig. 100 shows x as a function of t. Starting with a positive initial position x_{01}, the velocity will be negative; hence the motion is centered about an equilibrium position a units above the t-axis until it reaches the minimum x_{02}, where $|x_{02}| = |x_{01} - 2a|$.[2] From this point it proceeds with positive velocity so the motion is centered about an equilibrium position a units below the t-axis until it reaches the maximum x_{03}, where $|x_{03}| = |-x_{02} - 2a| = |x_{01} - 4a|$, etc. Thus the motion consists of damped oscillations

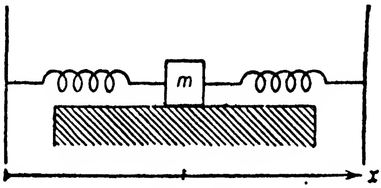


Fig. 99.

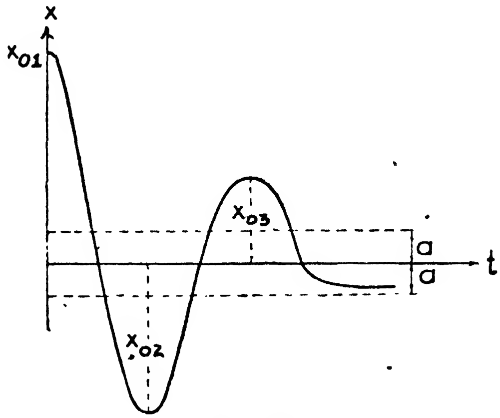


Fig. 100.

whose successive maxima form an arithmetic progression with difference $-4a$ (in contrast to the case of linear friction where they form a geometric progression). It is clear that eventually (after a finite number of oscillations) one of these maxima or minima will occur

[1] More precisely, no unique linear equation describes the process; otherwise we would have been able to represent by $m\ddot{x} + kx = f(\dot{x})$ not only this particular case but also more general laws of friction. The method of replacement of one non-linear equation by several linear equations very often simplifies the problem and makes it possible to obtain good quantitative results.

[2] The body can, of course, stop. Whether it stops or not depends on whether the maximum frictional force f_0 or the spring force is greater at the point x.

inside the strip $|x| < a$, when we have $v = 0$ and $|kx| < f_0$, and so the motion will then stop.

There are several noteworthy differences between a system with constant friction and a linear oscillator with dissipation (friction proportional to velocity). The first is the absence of anything resembling a logarithmic decrement in the case of constant friction; a natural consequence of the fact that the maxima form an arithmetical and not a geometrical progression. The second difference lies in the fact that with constant friction there is not even a quasi-period. For the time τ from a maximum to the subsequent zero is greater than that τ_1 between

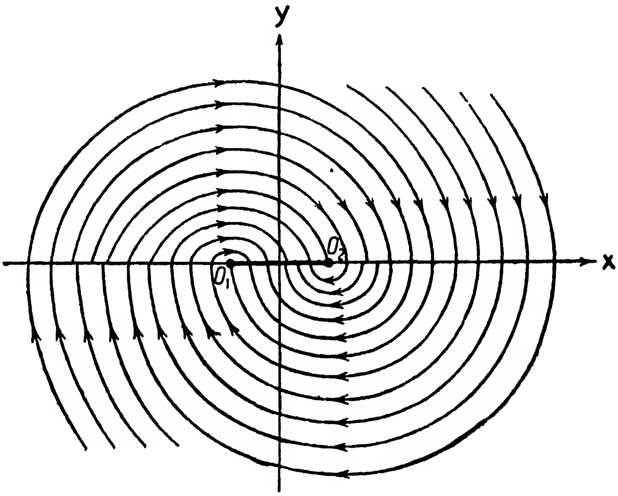


FIG. 101.

the zero and the next maximum, and τ increases as the maxima decrease. Finally in any system with constant friction one may always choose the initial conditions so that some oscillations take place before the system stops. Hence systems with constant friction cannot be classified, like the linear dissipative oscillators, into periodic and aperiodic systems. That oscillations do take place under suitable initial conditions is shown as follows. Starting at x_{01} with zero velocity the initial energy is the potential energy $V_1 = (kx_{01}^2)/2$. The work P_1 spent in resisting friction during the first "conditional" period is $P_1 = (x_{01} - x_{02})f_0$, while the potential energy is $V_2 = kx_{02}^2/2$. Since $V_1 - V_2 = P_1$, we find $x_{01} + x_{02} = 2f_0/k = 2a$. Since P_1 is a linear, and V_1 a quadratic function of x_{01}, we see that for large enough x_{01} the system will have energy to spare at the end of the first conditional period and therefore it will oscillate at the start.

Now consider the motion in the phase plane. Setting $y = \dot{x}$, the

equations are

$$y < 0: \qquad \frac{dy}{dx} = \frac{-\omega_0^2(x - a)}{y},$$

$$y > 0: \qquad \frac{dy}{dx} = \frac{-\omega_0^2(x + a)}{y}$$

whose integrals are

$$y < 0: \qquad \frac{(x - a)^2}{R_1^2} + \frac{y^2}{R_1^2\omega_0^2} = 1,$$

$$y > 0: \qquad \frac{(x + a)^2}{R_2^2} + \frac{y^2}{R_2^2\omega_0^2} = 1,$$

where R_1 and R_2 are constants of integration. Hence the paths are obtained by piecing together these half-ellipses, yielding the

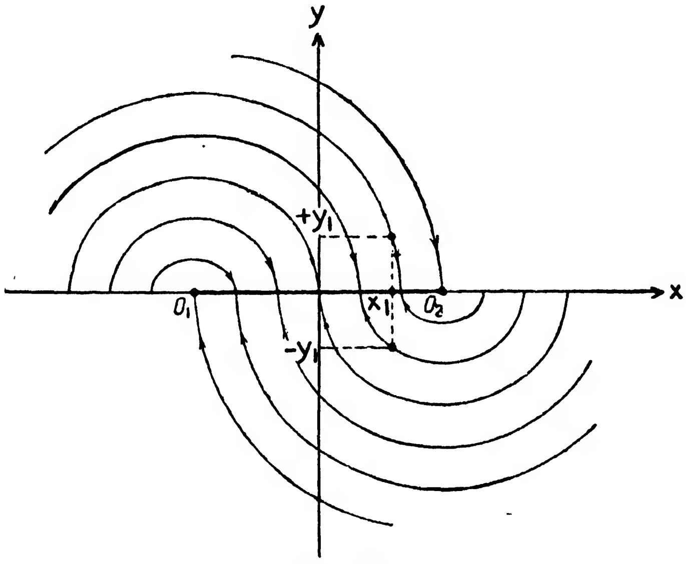


FIG. 102.

spirals shown in Fig. 101. The motion along these spirals is inward until the segment O_1O_2 (Fig. 102) is hit, where, as a reference to Fig. 102 shows, motion cannot continue. The points of O_1O_2 are the equilibrium positions. Now a point x_1 on O_1O_2 (Fig. 102) whose y-coordinate is changed by a small amount $\pm y_1$ ($y_1 > 0$), as shown in Fig. 102, returns to O_1O_2 farther from or nearer the origin than at the start according as the displacement is $+y_1$ or $-y_1$. Because the upper half ellipses are steeper, the point is brought near the origin by $-y_1$ by a greater amount than it is displaced away from it by $+y_1$. Hence, if the system is subjected to an equal number of such small impulses in

both directions, the net effect will be to move it toward the "true" equilibrium position, the origin.

The existence of an entire segment of equilibrium positions (called *stagnation*) and the approach to true equilibrium as a result of numerous small impulses is frequently utilized in measuring devices and in indicators subjected to dry friction. Stagnation is, of course, undesirable and does not occur with liquid friction. Hence for dry friction in bearings a certain device, called the Brown system, is used to change dry into liquid friction. It consists in giving the axis a constant motion back and forth along the bearing so that the friction between the axis and bearing becomes proportional to the velocity of rotation in the direction of rotation, provided, of course, that the velocity of rotation is sufficiently small. The system behaves then as if it were subjected to liquid and not to dry friction.

§4. Vacuum Tubes with Discontinuous Characteristic and Their Self-Oscillations

The characteristic which we have in mind is like the one of Fig. 105: zero to the left of the vertical axis, constant and non-zero to the right. A well-known instance is that of the mercury-arc rectifier. Here again, of course, the simplified form of the characteristic results from an extensive idealization to which we return in a moment. For the present it is clear that with the discontinuous characteristic we will have another occasion to apply the process of piecing together in a suitable way the solutions of distinct differential equations.

As regards the self-oscillations, they arise in principle in the following manner. Consider the tuned-plate oscillatory circuit of Fig. 103 in which energy is provided by a storage battery. Let the resistance be small so that the system naturally goes into a damped oscillation of period T. Suppose that the energy dissipated throughout the period is compensated by energy from the storage battery during the rising part of the oscillation, since it is only then that the battery is in effective operation. If the compensation is exact, i.e. if there is neither gain nor loss of energy, a prolonged oscillation will be reached. That is to say, the system will go into a steady oscillatory state with period T. These are the self-oscillations which our detailed treatment will bring out. It is the first instance of an all important phenomenon of which many others will be found in the sequel.

We start then with the basic circuit of Fig. 103 with a parallel-resonant circuit in series with the plate of the tube and the feedback coupling included in the grid circuit Neglecting the grid current and

plate reactance and denoting by x the current in the branch of the circuit containing the inductance, we have, with the notations indicated in Fig. 103,

$$L\dot{x} + Rx + \frac{1}{C}\int (x - i_p)\,dt = 0,$$

from which follows

$$L\ddot{x} + R\dot{x} + \frac{x}{C} = \frac{i_p}{C} = \frac{f(e_g)}{C}$$

where $i_p = f(e_g)$ is the transfer characteristic of the tube (the plate current as a function of the grid voltage). We idealize this relation as follows: we suppose that for $e_g > 0$ the plate current reaches instantaneously its maximum I_s and that for $e_g < 0$ it becomes instantaneously zero. This is tantamount to assuming for the characteristic the graph of Fig. 105. It is easily seen that, if the oscillations in the tube are sinusoidal, then the greater their amplitude the more closely would our assumption be fulfilled. For if the amplitude of the grid voltage is many times greater than the saturation voltage, then during the major part of the first half period there is no plate current and during the major part of the second half period the plate current is equal to the saturation current. The voltage on the grid is $e_g = M\dot{x}$, and hence the amplitude of e_g will be the greater the larger the amplitude of the sinusoidal current x and the mutual inductance M providing feedback coupling. Now as we shall see later, all other things being equal, the current x will be more nearly sinusoidal and its amplitude greater the smaller the damping of the system. Hence our idealization is physically significant when the damping is sufficiently small and the feedback coupling sufficiently strong.

Let the coils be so connected that $M > 0$. Thus $e_g > 0$ implies $\dot{x} > 0$, and conversely. As a consequence, and assuming the characteristic in accordance with Fig. 105, the oscillator will satisfy the following equations:

$$\ddot{x} + 2h\dot{x} + \omega_0^2 x = \omega_0^2 I_s, \qquad \dot{x} > 0 \tag{10}$$

$$\ddot{x} + 2h\dot{x} + \omega_0^2 x = 0, \qquad \dot{x} < 0, \tag{11}$$

where I_s is the non-zero value assumed by i_p. It is evident that $x(t)$ is not a solution of a linear equation with constant coefficients, i.e. we certainly do not have here a linear system. As already stated, we assume small damping and hence h small. As a consequence the two equations will have oscillating solutions. We must also keep in mind

that $\omega_0^2 = 1/LC$. It is clear that (11) has solutions of the form

$$(12) \qquad x = Ae^{-ht}\cos(\omega t + \alpha), \qquad \omega = \sqrt{\omega_0^2 - h^2}.$$

As for (10) we may write it

$$(x - I_s)^{\cdot\cdot} + 2h(x - I_s)^{\cdot} + \omega_0^2(x - I_s) = 0$$

and so its general solution is

$$(13) \qquad x = I_s + Be^{-ht}\cos(\omega t + \beta),$$

which represents an oscillation whose equilibrium position is I_s. Thus the motion is composed of oscillatory portions (13) for $\dot{x} > 0$ and of oscillatory portions (12) for $\dot{x} < 0$. We must now ask how

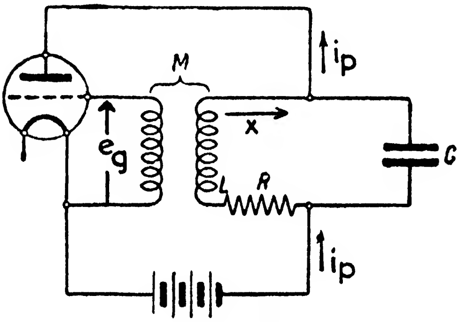


FIG. 103.

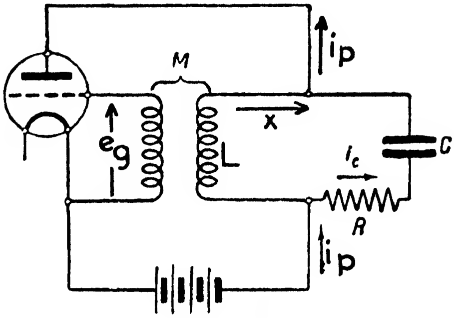


FIG. 104.

these are fitted together to form the true $x(t)$, the "motion" of the system.

The answer is found in the physical condition of the jump of Chapter I, §5. Referring to our circuit (Fig. 103), there must be no instantaneous change in the energy stored in any part of the system. Hence $\frac{1}{2}Lx^2$ and $\frac{1}{2}CV^2$ (V = the condenser potential difference) must be continuous. Hence V and x must be continuous. Since $L\dot{x} + Rx = V$, $\dot{x}$ must be continuous and the continuity of x and $\dot{x}$ is sufficient for that of V. (Notice that this conclusion would not follow from the circuit of Fig. 104.) Let us follow this up. Suppose that at first $\dot{x} > 0$ so that we follow a solution of (13). In Fig. 106 we are then on an ascending branch. This is to be followed till $\dot{x} = 0$, x maximum. Arrived there we can only continue with a descending arc, i.e. a solution of (12), since otherwise we would have a branch of (13), i.e. a sinusoidal curve with a horizontal inflection. The descending arc is now continued to its minimum, where a similar argument shows that it must be followed by an ascending arc (13), etc. Similarly, of course, if we were to start on a descending branch. Thus in all cases the solu-

tion $x(t)$ is pieced together by means of alternating solutions of (12) (descending arcs) and of (13) (ascending arcs) to form an oscillating graph as in Fig. 106. That both $x(t)$ and $\dot{x}(t)$ are continuous is obvious.

As we shall see, the system tends to a steady oscillatory state. To bring this out let x_1, x_2, x_3 be three consecutive extrema of $x(t)$: maximum, minimum, maximum. From x_1 to x_2 we have a solution

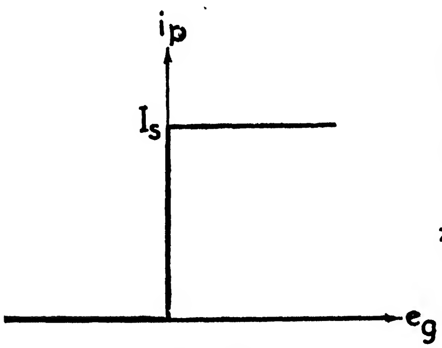


Fig. 105.

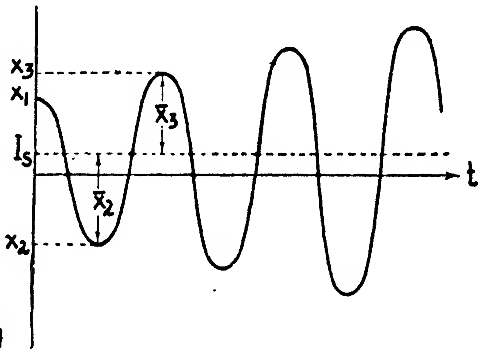


Fig. 106.

(12) with logarithmic decrement h and period $2T = 2\pi/\omega$, hence

$$-x_2 = x_1 e^{-hT}.$$

Similarly, since the median line for the ascending branch from x_2 to x_3 is $x = I_s$, we have

$$x_3 - I_s = (-x_2 + I_s)e^{-hT} = x_1 e^{-2hT} + I_s e^{-hT}$$

and hence

$$x_3 = x_1 e^{-2hT} + I_s(1 + e^{-hT}).$$

Set temporarily $e^{-2hT} = \lambda < 1$, $I_s(1 + e^{-hT}) = \mu$. Thus $x_3 = \lambda x_1 + \mu$. Consider successive maxima $x_1, x_3, \cdots, x_{2k+1}$ and write this relation for all consecutive pairs. Thus

$$\begin{aligned} x_{2k+1} &= \lambda x_{2k-1} + \mu \\ x_{2k-1} &= \lambda x_{2k-3} + \mu \\ &\cdots\cdots\cdots\cdots \\ x_3 &= \lambda x_1 + \mu. \end{aligned}$$

Multiplying by $1, \lambda, \cdots, \lambda^{k-1}$ we find

$$x_{2k+1} = \lambda^k x_1 + \mu(1 + \lambda + \cdots + \lambda^{k-1}).$$

Since $0 < \lambda < 1$, as $k \to +\infty$ the geometric progression $\to 1/(1 - \lambda)$

and $\lambda^k \to 0$. Hence $x_{2k+1} \to x_0$ where

$$x_0 = \frac{\mu}{1-\lambda} = \frac{I_s(1+e^{-hT})}{1-e^{-2hT}} = \frac{I_s}{1-e^{-hT}}. \tag{14}$$

In the same way it may be shown that x_{2k} tends to a value

$$\bar{x}_0 = -x_0 e^{-hT}.$$

Thus no matter what the initial situation, the current tends to be oscillatory with a steady amplitude (maximum) x_0; that is to say, we have a self-oscillation which takes place regardless of the initial

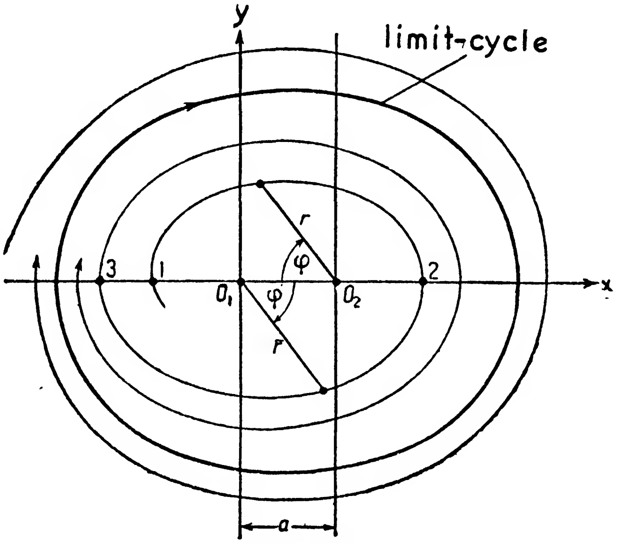


FIG. 107.

conditions, under however slight a perturbation of equilibrium, and depends solely upon the nature of the system.

The observation just made implies also that, if the system starts very near the steady state, then it tends towards that steady state. Hence the latter is stable. In other words, on the least provocation the circuit tends to assume a unique and well-defined oscillation. Notice that the period of this oscillation is still $2T$, the time between two maxima of the damped linear oscillations (12) or (13). The expression $d = hT$ is the logarithmic decrement of both oscillations, and we have

$$x_0 = \frac{I_s}{1-e^{-d}}.$$

The phase plane diagram is also instructive (Fig. 107). The system (12) gives rise in the whole plane to spirals asymptotic to the origin

and of which only the lower half must be preserved. The system (13) gives rise to spirals asymptotic to the point $x = I_s$, $y = 0$, i.e. to a point I_s units to the right of the origin on the x-axis and only the upper half of these spirals must be preserved. The sense of description is shown on the figure. The resulting paths are spiral curves which are asymptotic to one of them which is simply an oval and corresponds to the steady oscillations. It intersects the x-axis at x_0 and $\bar{x}_0$. Its upper half is part of a spiral corresponding to (13) and its lower half of a spiral corresponding to (12).

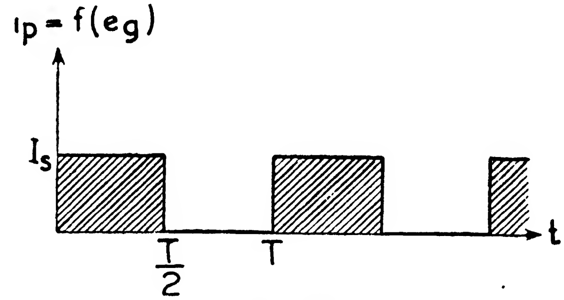


FIG. 108.

One may obtain a Fourier series for the self-oscillation $x(t)$ in the following manner. Since for half of the period $2T$ it satisfies (10) and for the other half (11), the right-hand sides are the values of the function $\omega_0^2 F(t)$ defined by

$$\begin{aligned} F(t) &= a, \qquad 0 < t < \frac{\pi}{\omega} \\ F(t) &= 0, \qquad \frac{\pi}{\omega} < t < \frac{2\pi}{\omega} \\ a &= I_s \end{aligned}$$

which form the square wave of Fig. 108 and $x(t)$ is the periodic solution of period $2\pi/\omega$ of

(15) $$\ddot{x} + 2h\dot{x} + \omega_0^2 x = \omega_0^2 F(t)$$

The expansion of $F(t)$ in Fourier series is the well-known series of period $2\pi/\omega$:

$$F(t) = a\left\{\frac{1}{2} + \frac{2}{\pi}\sum_{k=0}^{+\infty} \frac{\sin\,(2k+1)\omega t}{2k+1}\right\}.$$

Proceeding as if $F(t)$ were a finite sum, we find the special solutions for the equations corresponding to the individual terms of $F(t)$:

$$\ddot{x} + 2h\dot{x} + \omega_0^2 x = \omega_0^2 \frac{a}{2}$$

$$\ddot{x} + 2h\dot{x} + \omega_0^2 x = \frac{2\omega_0^2 a}{\pi} \frac{\sin (2k + 1)\omega t}{2k + 1}$$

By adding them we find

$$x(t) = a \left\{ \frac{1}{2} + \frac{2\omega_0^2}{\pi} \sum_{k=1}^{k=+\infty} (-a_k \cos (2k - 1)\omega t + b_k \sin (2k - 1)\omega t) \right\}$$

where

$$a_k = \frac{2h\omega}{c_k}, \qquad b_k = \frac{\omega_0^2 - \omega^2(2k - 1)}{(2k - 1)c_k}$$

$$c_k = (\omega_0^2 - \omega^2(2k - 1)^2)^2 + 4h^2\omega^2(2k - 1)^2.$$

It is not difficult to show that $x(t)$ represents the desired periodic solution of (15), i.e. the self-oscillation of the system. We may thus think of this oscillation as consisting of a certain fundamental harmonic and of its higher harmonics.

Let us write

$$x(t) = \tfrac{1}{2}C_0 + \sum_{k=1}^{+\infty} C_k \cos [(2k - 1)\omega t + \alpha_k]$$

where the C_k are the amplitudes. The number

$$1 + \rho^2 = \frac{\tfrac{1}{2}C_0^2 + \sum_{k=1}^{+\infty} C_k^2}{C_1^2}$$

is known as the *harmonic coefficient* of the system. The term is justified on the ground that the smaller ρ the closer $x(t)$ reduces to its fundamental harmonic $C_1 \cos (\omega t + \alpha_1)$.

Now a simple calculation shows that as $h \to 0$ every C_k except C_1 remains finite and $C_1 \to +\infty$. Hence $\rho \to 0$. Thus as $h \to 0$, i.e. as dissipation $\to 0$ ($R \to 0$), the system tends to act, when R is small, as if one had ordinary harmonic resonance, i.e. it tends to produce something akin to self-resonance.

§5. THEORY OF THE CLOCK

A clock is an oscillatory mechanism with sustained oscillations whose amplitude is independent of the initial conditions. To start the oscillations a rather strong initial impulse is required; failing this

the clock goes back to rest. Our theory will aim to explain all these details in full.

Roughly speaking, the mechanism includes: (a) an oscillatory system: horizontal or vertical pendulum, etc., referred to, for the sake of simplicity, as a pendulum; (b) a source of energy: weight or spring; (c) a control correlating the first two elements. For certain positions of the pendulum the control operates and allows the required energy to pass in the form of an impulse. The duration of the impulse varies from clock to clock, but in the good clocks it is quite short. The control operates generally twice per period, and this close to the position of equilibrium, i.e. where the velocity is greatest. An important feature is that the moment when the control operates depends solely on the position of the pendulum. Furthermore its action, and notably the magnitude of the impulse, depends on the state of the pendulum. Hence the whole operation depends on the position and velocity of the parts and not on the time. Thus a clock is an autonomous system.

For simplicity we shall assume that the control acts once in each period of the pendulum, producing an instantaneous change in velocity. We shall consider two "laws of impulse," one asserting that the change in velocity is constant, the other that the change in kinetic energy is constant. That is to say, if v_0, v_1 are the velocities of the pendulum just before and after the impulse, the first states that $mv_1 - mv_0 =$ constant, the second that $mv_1^2 - mv_0^2 =$ constant.[1] The second is very natural since it holds exactly when the unwinding mechanism consists of a weight which is displaced downward the same distance (thus doing the same work) in each period. Of the two assumptions the second is the more plausible since it asserts that each impulse contributes the same amount of energy. The first assumption may imply a larger change of energy for some impulses than for others; the lower the velocity before the impulse the less the energy transferred to the system. This assumption is not, however, entirely ruled out and so both assumptions will be discussed here.

In addition to the nature of the impulse there are also hypotheses to be made regarding friction. The simplest are: (a) linear friction, or friction proportional to the velocity; (b) constant friction. The two types lead to quite different results.

1. The clock with linear friction. We consider first the case where the change of velocity at each impulse is constant. The method here is similar to that for a vacuum tube oscillator with discontinuous

[1] This second assumption is the usual one in the theory of the clock. See, for example, J. Andrade, *Horlogerie et chronométrie*, Paris, 1924.

characteristic. Let the damping be assumed small and let d be the logarithmic decrement of the system. Let also a be the change of velocity produced by the control mechanism and y_1 the initial velocity (i.e. immediately following the first impulse). Then the velocities y_2 and y_3 just before and after the second impulse are

$$y_2 = y_1 e^{-d}, \qquad y_3 = y_1 e^{-d} + a.$$

For periodicity we must have $y_3 = y_1 = y_0$, y_0 being the stationary amplitude. Hence $y_0 = a/(1 - e^{-d})$. As with the vacuum tube, the stationary amplitude is stable, and no matter how small y_1 may be the oscillations grow. Thus our model is self-exciting and is therefore to be rejected since real clocks are not. All this is confirmed by the phase portrait[1] (Fig. 109). The representative point beginning at y_1 follows one of the spirals and returns to a position y_2 below y_1, then rises to $y_2 + a$ and proceeds generally on another spiral. Since $y_1 - y_2$ increases from 0 to $+\infty$ when y_1 increases from 0 to $+\infty$, $y_1 - y_2$ passes exactly once through the value a. Then and only then the motion continues on the first spiral and the representative point describes a closed curve, i.e. we have an oscillation. Fig. 109 shows that it is stable and self-exciting.

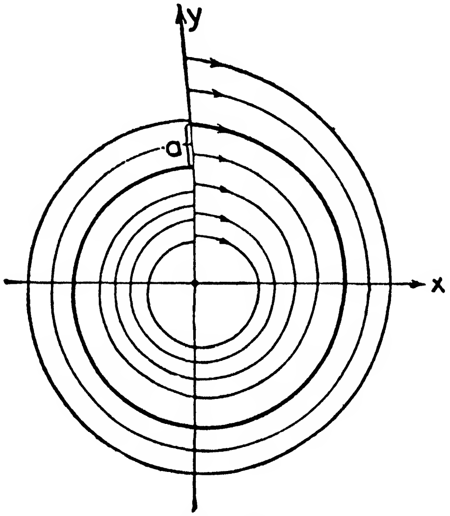


FIG. 109.

We now examine the second law of impulse which reduces here to

$$y_3^2 - y_2^2 = h^2 = \text{constant}.$$

This time we find

$$y_2 = y_1 e^{-d}, \qquad y_3 = \sqrt{y_2^2 + h^2} = \sqrt{y_1^2 e^{-2d} + h^2}.$$

The stationary amplitude $\bar{y}$ is determined by $y_3 = y_1 = \bar{y}$. Hence

$$\bar{y}^2(1 - e^{-2d}) = h^2, \qquad \bar{y} = \frac{h}{\sqrt{1 - e^{-2d}}}.$$

[1] We assume that the impulse takes place once in a period and for maximum positive velocity.

The situation is found to be the same as before: unique oscillation with self-excitation. The only difference is that a is not fixed, its value being

$$a = \sqrt{y_2^2 + h^2} - y_2,$$

but otherwise nothing is changed. Thus this model must also be rejected. It follows that linear friction is not admissible.

2. The clock with constant friction. As we have seen, in suitable units the equation of motion of a linear oscillator with constant friction

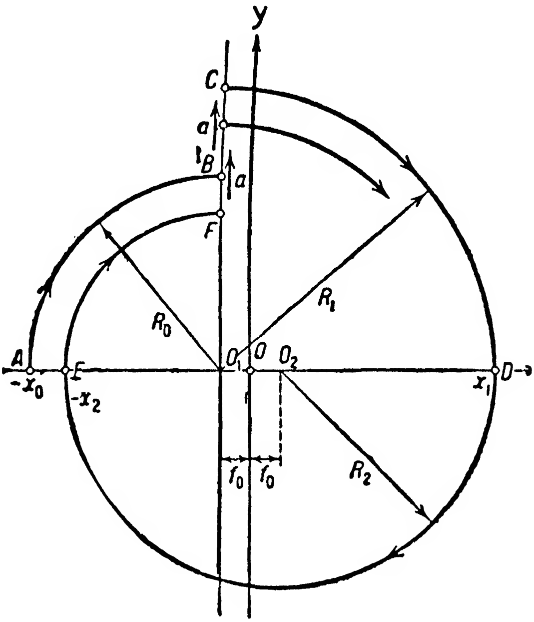


FIG. 110.

f_0 per unit mass is

$$\begin{aligned} \ddot{x} + x &= -f_0 \quad &\text{for} \quad \dot{x} > 0 \\ \ddot{x} + x &= +f_0 \quad &\text{for} \quad \dot{x} < 0. \end{aligned}$$

We have seen that the paths are spirals formed from half-ellipses—actually half-circles here because of the choice of units. For simplicity we assume that the impulse occurs at $x = -f_0$ instead of $x = 0$. The first impulse assumption that $mv_1 - mv_0 =$ constant becomes, in terms of the coordinates $y_1 - y_0 = a$. Now consider the paths in the phase plane. A representative point starting at $A(-x_0,0)$ follows the path indicated in Fig. 110. The center of the upper half circles is $(-f_0,0)$, that of the lower ones is $(f_0,0)$. Starting on a circle of radius R_0, the point changes to a circle of radius $R_0 + a$ after 90°, cuts the

positive x-axis at distance $R_0 + a - 2f_0$ from $(f_0,0)$, then follows a circle of radius $R_0 + a - 2f_0$ until it returns to the negative axis at distance $R_0 + a - 4f_0$ from $(-f_0,0)$. Thus the character of the motion depends on the sign of $a - 4f_0$.

Case I: $a - 4f_0 < 0$. As shown in Fig. 111 the oscillations will be damped out when, after a finite number of strokes, the point gets into the equilibrium region: $-f_0 < x < f_0$.

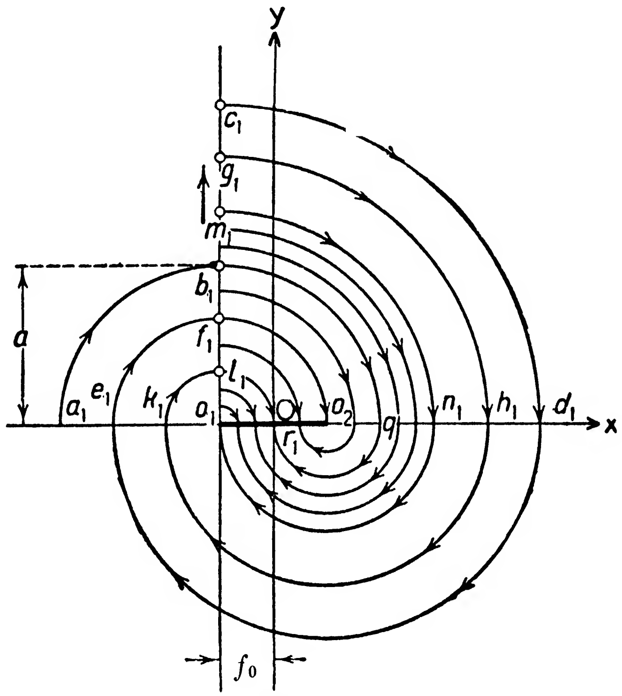


FIG. 111.

Case II: $a - 4f_0 > 0$. This is shown in Fig. 112. Points starting in the shaded region will reach equilibrium after a finite number of strokes. For points starting in the rest of the plane (including the boundary curve) the oscillations of the system will grow infinitely.

Case III: $a - 4f_0 = 0$. This is shown in Fig. 113. Points starting in the shaded region will reach the equilibrium state before describing a full revolution. For points starting outside the shaded region all motions will be periodic with the amplitude depending on the initial conditions. Here we have a continuum of periodic motions and instability with respect to small changes of the parameters; any change of f_0 throws us into Case I or II. Both properties are characteristic of conservative systems.

Thus the assumption of constant friction with the impulse law, $mv_1 - mv_0 =$ constant, does not correspond to actual clocks. We shall now assume constant friction and the impulse law, $mv_1^2 - mv_0^2 =$ constant, and show that this combination is satisfactory. The impulse law gives $y_1^2 - y_0^2 = h^2$, where h is a constant, hence $a = \sqrt{y_0^2 + h^2} - y_0$. (See Fig. 114.) The phase portrait is shown in Fig. 115. A

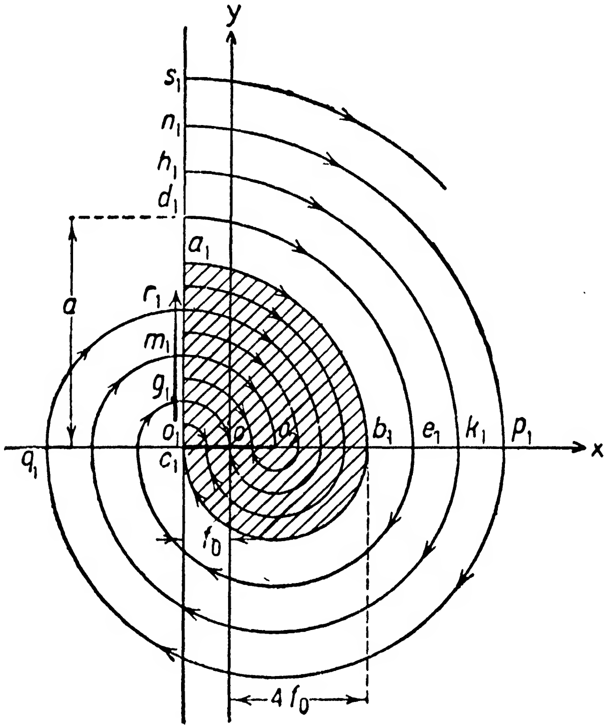


FIG. 112.

representative point starting at $(-x_0,0)$ on a circle of radius R_0, center $(-f_0,0)$, moves after 90° to a circle of radius R_1 where $R_1^2 - R_0^2 = h^2$, then cuts the positive x-axis at $(x_1,0)$ where $x_1 = R_1 - f_0$. Then it follows a circle of radius $R_2 = R_1 - 2f_0$, center $(f_0,0)$, cutting the negative x-axis at $(-x_2,0)$ where $x_2 = R_1 - 3f_0$. Hence

$$(x_2 + 3f_0)^2 - (x_0 - f_0)^2 = h^2.$$

For a periodic motion we have $x_2 = x_0$; hence, setting $x_2 = x_0$ in this equation and solving for x_0, we find

$$x_0 = \frac{h^2}{8f_0} - f_0.$$

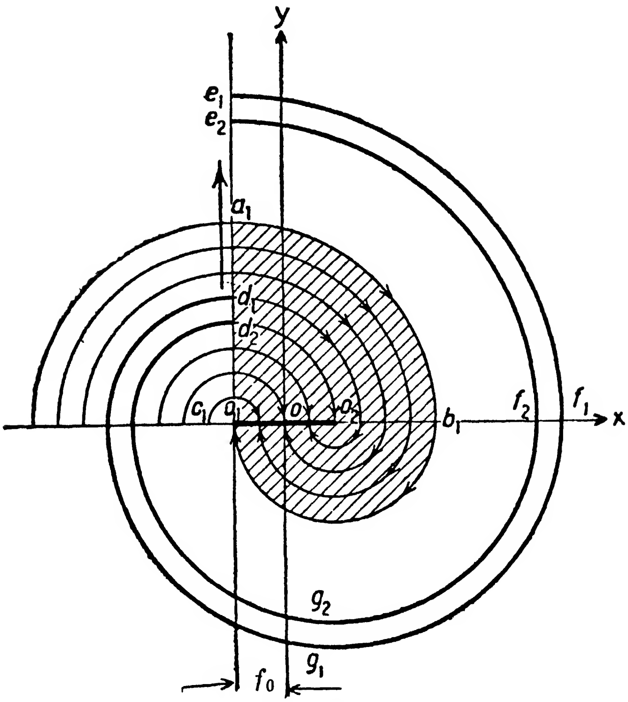


Fig. 113.

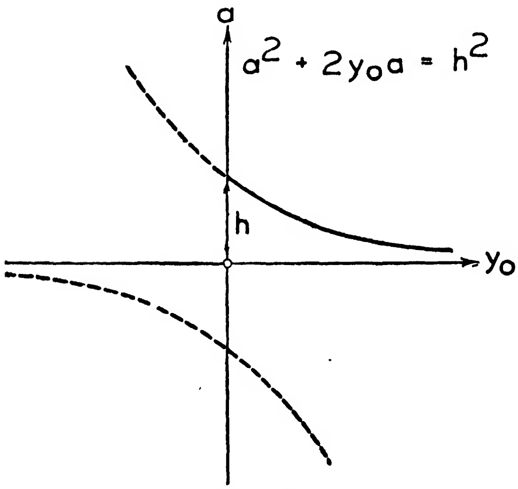


Fig. 114.

To have a periodic motion we need $x_0 > f_0$, or $h^2 > 16f_0^2$, since otherwise $(-x_0,0)$ will lie on the segment of equilibrium states. This imposes a condition on the energy source demanding it to be of greater strength for larger f_0. If this condition is satisfied, then we have a

unique periodic motion of definite amplitude. The closed path consists of parts of circles and a segment of length a on $x = -f_0$. It follows easily from a theorem of Koenigs[1] that the neighboring motions

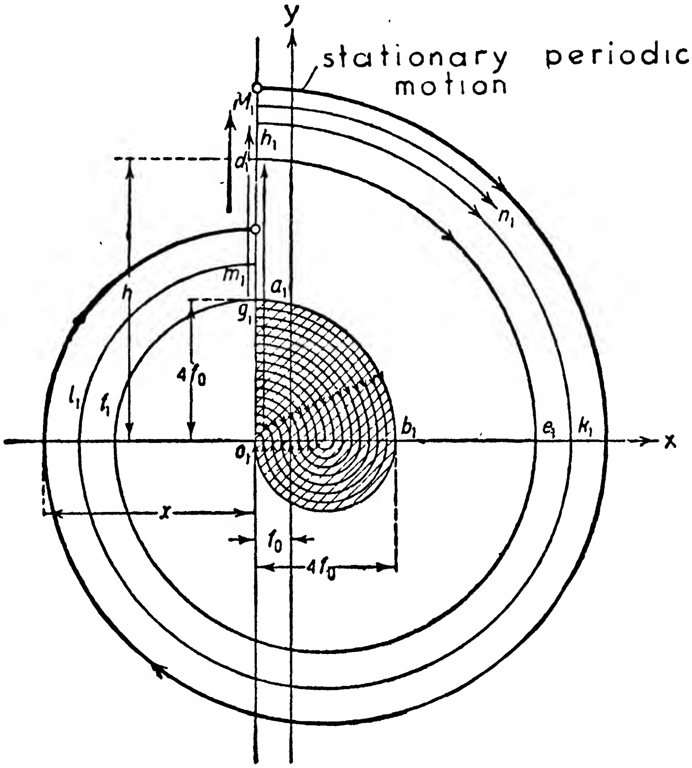


FIG. 115.

approach the periodic motion, i.e. the periodic motion is stable. This model clearly has the two essential features of a clock: existence of a

[1] The theorem of Koenigs, of interest in the theory of oscillations, is as follows: Let $F(x,y)$ be analytical near $(\bar{u},\bar{u})$, where $F(\bar{u},\bar{u}) = 0$. Let us suppose that a sequence $u_1,u_2, \cdots$ is such that $F(u_n,u_{n+1}) = 0$ for all n. Then the sequence $\rightarrow \bar{u}$ if: (a) the points u_n from a certain n are sufficiently near $\bar{u}$, and (b) the slope $-F_x/F_y$ of $F(x,y) = 0$ at $(\bar{u},\bar{u})$ is <1 in absolute value. If (a) holds but the slope >1 in absolute value, then the sequence does not converge to $\bar{u}$. In our case with the sequence $x_0,x_2,x_4, \cdots$, and $F \equiv (y + 3f_0)^2 - (x - f_0)^2 - h^2$, it is found that the absolute slope is

$$\left|\frac{\bar{x} - f_0}{\bar{x} + 3f_0}\right| < 1,$$

and that property (a) holds. Hence we have convergence to $\bar{x}$. For further details see Koenigs, Recherches sur les équations fonctionnelles, *Bulletin des Sciences Mathématiques*, 1883.

unique stable stationary amplitude and necessity of an initial impulse of a certain magnitude to start the oscillation. The greater the friction the greater must be the initial impulse.

The friction in an actual clock is partly of constant type, as in that from the control, and partly variable, as in that from the air resistance of the pendulum. If we consider a model containing both types, assuming the variable friction linear, we get nothing really new. The image in the phase plane is simply made up of parts of spirals instead of circles. Constant friction is related to an essentially new property: no self-excitation and necessity of a strong initial impulse to start the periodic process.

Whenever an oscillatory system requires a minimum initial impulse, so that there are both a periodic motion and stable equilibrium, we shall say that we have *hard operating conditions*. *Soft operating conditions* are those in which periodicity is either impossible or else takes place whatever the initial conditions (on the least provocation), so that the system is self-exciting. Here there is only one stable state at a time, and it can be either periodic or an equilibrium.

3. A digression regarding vacuum tubes with discontinuous characteristics. For a clock, hard operating conditions arise from dry friction, while linear friction gives soft operating conditions. Another case of soft conditions is the vacuum tube oscillator of §4, with a characteristic composed of straight segments. Here, as we shall see, we have softness because the characteristic of the tube is not displaced, i.e. its vertical part passes through the point $e_g = 0$. Otherwise the working conditions become hard. The displaced discontinuous characteristic is a satisfactory idealization when the alternating voltage on the grid appreciably exceeds the saturation voltage of the tube and the operating point is displaced either into the saturation region or the region where the plate current is zero (i.e. a grid-bias voltage is applied). When the discontinuous characteristic is displaced, the equation of motion is the same as without displacement:

$$\begin{aligned} \ddot{x} + 2h\dot{x} + \omega_0^2 x &= a \qquad \text{for} \qquad \dot{x} > b \\ \ddot{x} + 2h\dot{x} + \omega_0^2 x &= 0 \qquad \text{for} \qquad \dot{x} < b. \end{aligned}$$

The phase portrait (Fig. 116) is obtained by cutting the phase plane along the line $y = b$ and shifting the upper half to the right by a. By continuity we may show again that there is one and only one closed path, made up of two half-spirals. All the paths except those in the shaded region tend to this one, while those in the shaded region tend to a stable equilibrium (a focus). Clearly a periodic process will

occur only if the initial voltage and current intensity are sufficiently great. We shall show that an idealization like that made for the clock —with instantaneous impulses from an apparatus of unlimited power— is satisfactory here also.

Neglecting as usual the effect of grid current and changes in plate voltage, we shall assume that the L_1CR oscillating circuit is in the grid

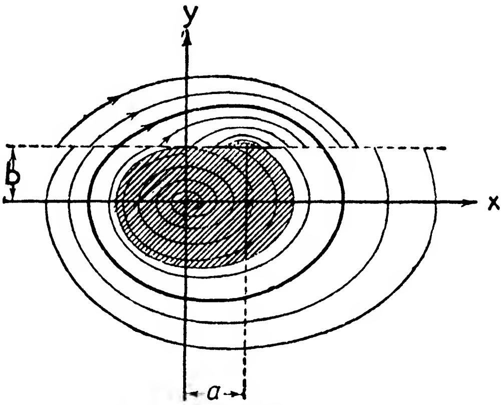


FIG. 116.

circuit (see Fig. 117). The oscillations will not depart far from the sinusoidal form if R is made small and C is made large so that the energy stored by L_1 and C is large compared to the energy dissipated by R during each period and the energy furnished by the battery during each period. The almost-sinusoidal oscillations are so large that the voltage of the grid enters far into the region of zero plate current and region of saturation. The grid voltage v changes sign twice during each period. When v passes through zero with positive slope, the plate current i_p rapidly (but not instantly) increases from zero to I_s (Fig. 105). As a result of the mutual inductance M between the coils for L_1 and L_2 this rapid change of current through L_2 causes an induced voltage $f(t) = M\dot{i}_p$ in the coil for L_1 which is effectively in series with the inductance L_1 and therefore with C and R. The induced voltage $f(t)$ is non-zero for such a short time that the charge q in C (and the voltage v across C) does not change appreciably. Also the voltage across R is small compared to $f(t)$. Therefore $f(t)$

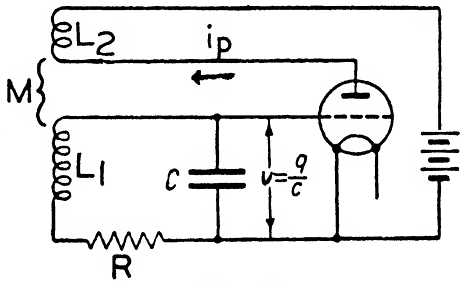


FIG. 117.

must appear almost entirely across L_1. In a similar manner when v passes through zero with negative slope, the plate current changes rapidly from I_s to 0. Then the induced voltage $f(t)$ occurs in the form of a short pulse across L_1 of opposite sign from the previous pulse (see Fig. 117*a*). If we idealize $f(t)$ to the form of an instantaneous pulse, then the situation is analogous to our idealization of the clock.

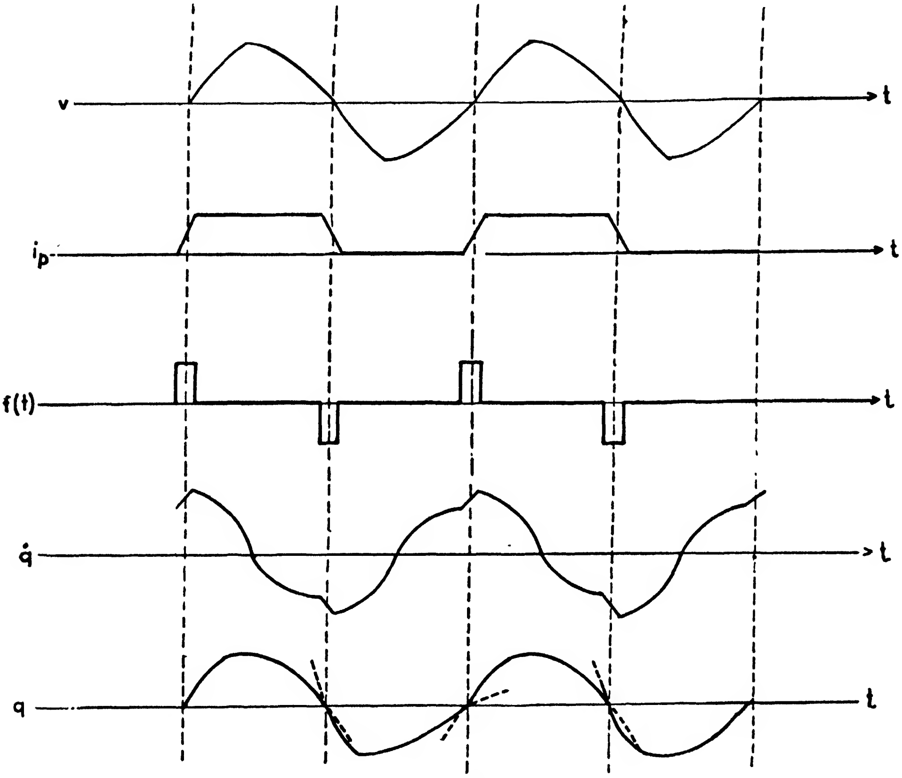


FIG. 117*a*.

The induced voltage $f(t)$ applied to L_1 over a short time interval will produce a jump $\Delta\dot{q}$ (see Fig. 117*a*) in the current $\dot{q}$ through L_1 (L_1 and C have the same current). The value is obtained from

$$L\frac{d\dot{q}}{dt} = f(t)$$

$$\Delta\dot{q} = \frac{1}{L}\int_t^{t+\tau} f(t)\,dt = \frac{M}{L}\int_t^{t+\tau} \dot{i}_p dt$$

where the interval t to $t + \tau$ is the time during which the plate current i_p changes. For the case when v passes through zero with

positive slope we have

$$\Delta\dot{q} = \frac{M}{L}(i_p(t+\tau) - i_p(t)) = \frac{M}{L} I_s.$$

Similarly when v passes through 0 with negative slope

$$\Delta\dot{q} = -\frac{M}{L} I_s.$$

Since $\Delta(L\dot{q})$ does not depend on the behavior of i_p in the interval $(t, t+\tau)$, but only on I_s, it is unimportant whether the jump is instantaneous or very rapid. We assume it instantaneous, and the equations of the system become

$$(16) \quad \begin{cases} \ddot{q} + 2h\dot{q} + \omega_0^2 q = 0 & \text{for} \quad q \neq 0 \\ \Delta\dot{q} = \dfrac{M}{L} I_s & \text{for} \quad q = 0,\ \dot{q} > 0 \\ \Delta\dot{q} = -\dfrac{M}{L} I_s & \text{for} \quad q = 0,\ \dot{q} < 0 \end{cases}$$

with the added condition that q varies continuously. Then the oscillogram of any motion of the system consists of arcs of damped sinusoids

$$q = Ae^{-h(t-t_0)} \cos(\omega(t - t_0) + \phi),$$

joining points of the time axis. Where two arcs join there is a difference in slope depending on the jump. This system is just like the clock where the impulses produce a constant change of velocity. The only difference is that q changes twice, instead of once in each period.

One may show, if need be, by examining the paths, that every solution of the system (16) tends to the periodic motion representing a stable self-oscillating process:

$$q = (-1)^n A e^{-h\left(t - t_0 - \frac{nT}{2}\right)} \sin \omega \left(t - t_0 - \frac{nT}{2}\right),$$

$$t_0 + \frac{nT}{2} \leqslant t \leqslant t_0 + \frac{n+1}{2} T,$$

$$A = \frac{MI_s}{L\omega} \frac{1}{1 - e^{-hT/2}}, \qquad \omega = \sqrt{\omega_0^2 - h^2} = \frac{2\pi}{T}.$$

Here n is an integer and t_0 an arbitrary phase. For small logarithmic decrements, $d = hT$, we have approximately:

$$A = \frac{2MI_s}{\omega_0 L d}.$$

As with an oscillator with a discontinuous characteristic, when dissipation $\to$ 0 the oscillations tend to sine waves in the sense that the harmonic factor $\to$ 0.

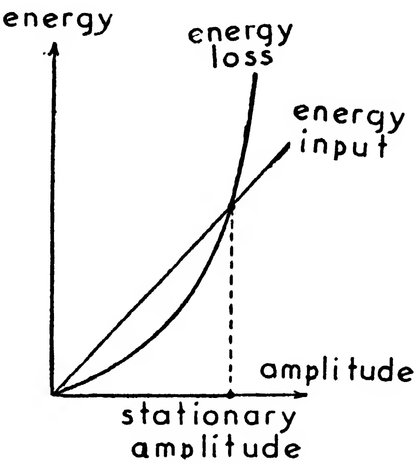


FIG. 118.

4. Return to the theory of the clock. We shall now explain some of our earlier conclusions by simple energy considerations. Recall that for linear friction the energy dissipated during a period is proportional to the square of the amplitude and that for constant friction it is linear. For the impulse law $v_1 - v_0 = \text{constant} = \Delta v_0$, the increase in energy is

$$\frac{m}{2}(v_0 + \Delta v_0)^2 - \frac{m}{2}v_0^2 = \frac{m}{2}(2v_0\Delta v_0 + \Delta v_0^2),$$

which is linear in v_0. For the impulse law $mv_1^2 - mv_0^2 = \text{constant}$, the increase in energy is constant. Our fundamental results are now clear.

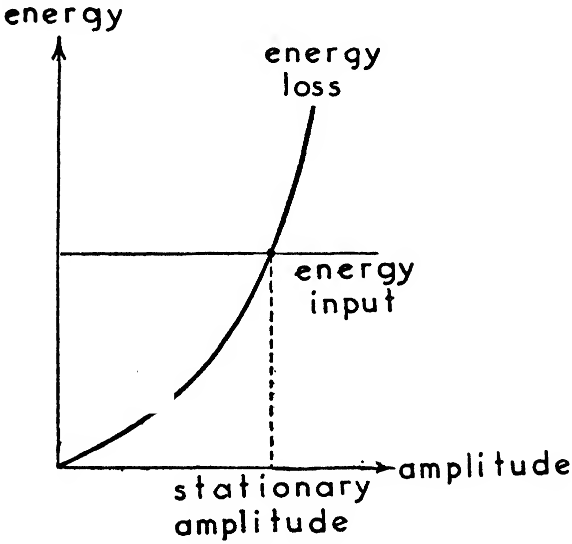


FIG. 119.

A periodic process is possible only when the energy of the system is the same at the end and beginning of a period. Consider whether this holds in our cases. In the first case (linear friction and impulse law

$v_1 - v_0 =$ constant) the losses of energy are proportional to the square of the amplitude, while the energy input is linear. Input and output are equal for only one amplitude, hence there is only one stationary amplitude (Fig. 118). In the second case (linear friction and impulse law, $v_1^2 - v_0^2 =$ constant) the loss of energy is quadratic and the input constant, so again there is only one stationary amplitude (Fig. 119). In the third case (constant friction and impulse law, $v_1 - v_0 =$ constant) loss and input are both linear, so there are either no stationary amplitudes or infinitely many. In the fourth case (constant friction

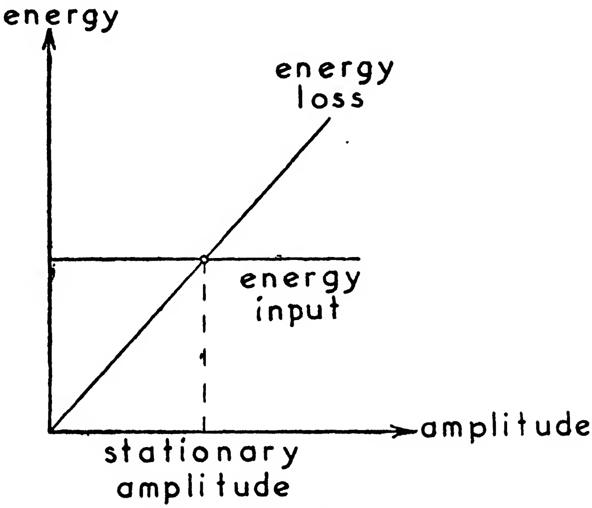


FIG. 120.

and impulse law, $mv_1^2 - mv_0^2 =$ constant) the loss is linear and input constant, so there is only one stationary amplitude (Fig. 120).

§6. Self-oscillatory Systems

We have just considered several systems, clocks and electronic circuits, which can be placed in a common class, the class of self-oscillatory systems. This class is all important in practice and its characteristic property is that there are produced perfectly definite oscillations which are on the whole independent of the initial conditions, and are intrinsic properties of the systems. The class includes, for instance, electric bells, arc oscillators, various buzzers, wind and string musical instruments, etc.

In a self-oscillatory system, particularly in an autonomous system (not explicitly dependent on the time), the energy, at least as regards the system itself, must also be provided in some "timeless" manner. Thus in general the energy will be provided in constant manner: by a

storage battery, for instance. However, as the work accomplished by the source of energy will generally depend upon the system, it may act periodically. Thus *a self-oscillatory system generates a periodic process from a non-periodic source.* From this standpoint a steam engine may be thought of as a self-oscillatory system.

§7. PRELIMINARY INVESTIGATION OF APPROXIMATELY SINUSOIDAL SELF-OSCILLATIONS

Self-oscillatory systems with one degree of freedom are described by an equation

$$\ddot{x} + \omega_0^2 x = F(x,\dot{x}) - 2h\dot{x} = f(x,\dot{x}). \tag{17}$$

An oscillating circuit with linear damping always leads to such an equation.[1] For a vacuum tube oscillator this holds with $\omega_0^2 = 1/LC$, $2h = R/L$ and $F(x,\dot{x})$ will then be the e.m.f. due to feedback coupling. There is compensation of the ohmic loss and hence a periodic process may arise. For the present we shall assume $F(x,\dot{x})$ and hence also $f(x,\dot{x})$ analytic over the whole phase plane.

The fundamental problem for the sequel is to determine whether self-oscillations are possible and if so to find, if only approximately, the spectrum of self-oscillations. Before taking up the general problem itself it will be found very much worth while to discuss approximately sinusoidal types.[2]

Let us suppose that the closed path in the phase plane representing a periodic solution lies outside a circle of fixed radius R_0. Then if $f(x,\dot{x})$ is sufficiently small outside the circle the periodic process will be approximately sinusoidal.[3] However, the clock and the vacuum

[1] A noteworthy special case is where $F(x,\dot{x})$ does not depend on x, so that instead of (17) we have $\ddot{x} + \omega_0^2 x = \psi(\dot{x})$. Note also that (17) can be written $\ddot{x} = G(x,\dot{x}) = F(x,\dot{x}) - 2h\dot{x} - \omega_0^2 x$. From the mathematical point of view the expression at the right is not unique but arises out of physical considerations.

[2] Although self-oscillations in their physical nature differ substantially from oscillations of conservative systems, the *form* of stabilized self-oscillations can differ as little as we choose and even coincide with the form of oscillations in a conservative system. In particular, in certain important cases in practice, they differ very little from those of a linear oscillator. Thus examining the oscillogram of the oscillations of an oscillator with discontinuous characteristic in the case of small h, it cannot be distinguished from an oscillogram of a harmonic oscillator.

[3] To prove this assertion consider the system:

$$\frac{dx}{dt} = y; \qquad \frac{dy}{dt} = -x + \phi(x,y),$$

which upon eliminating y reduces to the form (17). Assuming that this system

tube oscillator with discontinuous characteristic have shown that the smallness of $f(x,\dot{x})$ is not a necessary condition for approximately sinusoidal performance.

We recall some facts about resonance. In §4 we considered a linear oscillator subject to an external periodic force and with friction proportional to velocity. We considered it in a state close to one of its proper oscillations. By "close" is meant here that the period of the motion is close to the period of the oscillator and the harmonic coefficient is small. Now consider an external periodic force $\Phi(t)$, of period $2\pi/\omega$, acting on a harmonic oscillator with linear damping, whose proper frequency is also $2\pi/\omega$. Thus we have

$$\ddot{x} + \omega^2 x = \Phi(t) - 2h\dot{x}. \tag{17'}$$

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has a periodic motion, whose path lies outside the circle of radius R_0, and further that outside the circle R_0, $|\phi(x,y)| < \epsilon R_0$ where $\epsilon < \frac{1}{2}$, we obtain, when passing to polar coordinates r,θ:

$$\dot{\theta} = -1 + \frac{x\phi(x,y)}{r^2}, \qquad \dot{r} = + \frac{y\phi(x,y)}{r}.$$

Outside the circle of radius R_0

$$\left|\frac{x\phi(x,y)}{r^2}\right| < \left|\frac{\phi(x,y)}{R_0}\right| < \epsilon < \frac{1}{2}$$
$$\left|\frac{y\phi(x,y)}{r}\right| < |\phi(x,y)| < \epsilon R_0.$$

Let us estimate the "correction" of the period of the harmonic oscillator. We have

$$\int_0^{2\pi} \dot{\theta}(t)\,dt = -2\pi + a, \qquad |a| < 2\pi\epsilon;$$
$$\int_0^{\tau} \dot{\theta}(t)\,dt = -2\pi,$$

where τ is the unknown period. Hence

$$\left|\int_{2\pi}^{\tau} \dot{\theta}(t)\,dt\right| < 2\pi\epsilon;$$

and therefore

$$|\tau - 2\pi| < 4\pi\epsilon.$$

Regarding the maximum variation Δr of the radius-vector during the period we have

$$\Delta r < \int_0^{\tau} |\dot{r}_{\max}|\,dt < \int_0^{\tau} \epsilon R_0\,dt < \epsilon R_0(2\pi + 4\pi\epsilon).$$

It follows then that the closed path in question lies between two concentric circles whose difference of radii $< R_0(2\pi\epsilon + 4\pi\epsilon^2)$. It is obvious that, if we knew in advance that the path of the periodic motion is between the circles of radii R_0 and R_1 ($R_1 > R_0$), it would suffice to require the smallness of $\phi(x,y)$ only in the region between the two circles.

Let us set

$$\Phi(t) = P \cos \omega t + Q \sin \omega t + G(t),$$
$$G(t) = \frac{P_0}{2} + \sum_{n=2}^{+\infty} (P_n \cos n\omega t + Q_n \sin n\omega t),$$

where in $\Phi(t)$ the first two terms are the resonance terms.[1] Then there is a proper oscillation

$$x_1(t) = \frac{P \sin \omega t - Q \cos \omega t}{2h\omega} \tag{18}$$

where the resonance of the external force is compensated by the friction. For small h (when $P^2 + Q^2 \neq 0$) the system will develop under the action of $\Phi(t)$ a periodic motion as near as we please to (18) in the sense that it will dominate arbitrarily strongly the other harmonics of the oscillation or, if we please, will have a very small harmonic coefficient. Letting $x(t)$ be the precise solution of (17′) and defining $z(t)$ by: $x(t) = x_1(t) + z(t)$, it is clear that $z(t)$ is induced by the non-resonance term $G(t)$ and satisfies

$$\ddot{z} + \omega^2 z = G(t) - 2h\dot{z}.$$

In the following $z(t)$ will denote "forced" solutions of this equation. The harmonic coefficient ρ is given by

$$\rho^2 = \frac{\omega/\pi \int_0^{\frac{2\pi}{\omega}} z^2(t)\, dt}{\left(\frac{P^2 + Q^2}{4h^2\omega^2}\right)}$$

From the Fourier series solution for $z(t)$, which is readily calculated, we infer that

$$\frac{\omega}{\pi} \int_0^{\frac{2\pi}{\omega}} z^2(t)\, dt < \frac{1}{\omega^4} \frac{\omega}{\pi} \int_0^{\frac{2\pi}{\omega}} G^2(t)\, dt,$$

and hence that

$$\rho^2 \leqslant \frac{\int_0^{\frac{2\pi}{\omega}} G^2(t)\, dt}{P^2 + Q^2} \cdot \frac{4h^2}{\pi\omega}.$$

[1] That is, the constants P and Q are such that

$$\int_0^{\frac{2\pi}{\omega}} G(t) \cos \omega t\, dt = 0, \qquad \int_0^{\frac{2\pi}{\omega}} G(t) \sin \omega t\, dt = 0.$$

Hence the condition of smallness of the harmonic coefficient will be:

$$\int_0^{\frac{2\pi}{\omega}} \frac{G^2(t)\,dt}{\pi\omega} \ll \frac{P^2 + Q^2}{4h^2}.$$

With a given $\Phi(t)$, whatever its spectrum, for h sufficiently small and $P^2 + Q^2 \neq 0$, the harmonic coefficient can be made arbitrarily small.

We are interested in the case where the force is included in the system, and not external. The equation of motion is

$$\ddot{x} + \omega_0^2 x = F(x,\dot{x}) - 2h\dot{x}. \tag{19}$$

If the periodic motion of the self-oscillating process is $x = \phi(t)$, $\dot{x} = \dot{\phi}(t)$, then this satisfies

$$\ddot{x} + \omega_0^2 x = F(\phi(t),\dot{\phi}(t)) - 2h\dot{x} \tag{20}$$

which is the equation of a system acting under a force which depends on the time.[1] Hence self-oscillations can be considered as forced oscillations produced by an external force whose form depends on the nature of the self-oscillations. If the function of time $F(\phi(t),\dot{\phi}(t))$ satisfies the resonance conditions and its period is close enough to $2\pi/\omega_0$, then we can speak of self-resonance.[2]

The form of (20) is evidently not unique. Thus we can also write it as

$$\ddot{x} + \omega^2 x = F(\phi,\dot{\phi}) + (\omega^2 - \omega_0^2)\phi - 2h\dot{x} \tag{21}$$

where ω is the frequency. Then we would examine the external force

$$F_1(\phi,\dot{\phi}) = F(\phi,\dot{\phi}) + (\omega^2 - \omega_0^2)\phi$$

acting on the linear oscillator with another ("corrected") frequency. Sometimes in the form (20) the conditions of resonance are not fulfilled, while for a proper choice of ω they are fulfilled by (21).

[1] Let us note that equation (20) is only applicable to the periodic motion under consideration. In general, other periodic motions defined by (19) do not satisfy equation (21). Consequently, the solutions of the non-autonomous equation (20) do not enable us, for example, to investigate the stability of the periodic system.

[2] In particular, the notion of self-resonance enables one to conclude that, whenever $f(x(t),\dot{x}(t))$ in (17), as a function of time, is practically independent of the character of the oscillations in the circuit (for example, upon the magnitude of the amplitude), and its period tends to the period of the harmonic oscillator $2\pi/\omega_0$ when the damping of the circuit decreases, then we obtain more and more nearly sinusoidal motions when the damping decreases. This remark has practical interest.

By means of self-resonance and *assuming* that (19) has an approximately sinusoidal periodic solution, we obtain approximate expressions for the amplitude and frequency of the fundamental tone. Let the periodic solution of (19) be close, in the sense of small harmonic coefficient, to the sinusoidal solution

$$x_0(t) = A \cos \omega t, \qquad \dot{x}_0(t) = -A\omega \sin \omega t$$

with A and ω undetermined constants. Replacing the precise solution by a "zero approximation" $x_0(t) = A \cos \omega t$ in (21), we can again study the self-oscillations as forced oscillations. Then we obtain the equation

$$\ddot{x} + \omega^2 x = F_1(A \cos \omega t, -A\omega \sin \omega t) - 2h\dot{x}. \tag{22}$$

Expanding F_1 in Fourier series we may write

$$F_1(A \cos \omega t, -A\omega \sin \omega t) = P(A) \cos \omega t + Q(A) \sin \omega t + G(A,t)$$

where $P(A)$ and $Q(A)$ are the usual Fourier coefficients. The forced solution of (22) is

$$x_1(t) = \frac{P(A) \sin \omega t - Q(A) \cos \omega t}{2h\omega} + z_1(A,t)$$

where $z_1(A,t)$ contains the terms induced by the non-resonance term $G(A,t)$. For given ω, $P(A)$ and $Q(A)$ there is a proper oscillation

$$\frac{P(A) \sin \omega t - Q(A) \cos \omega t}{2h\omega}$$

and the resonance terms of the external force are compensated by the friction in these oscillations. Hence we can identify this proper oscillation with $x_0(t) = A \cos \omega t$, which, by assumption, produces the external force. This gives the equations[1]

$$P(A) = 0, \qquad Q(A) + 2h\omega A = 0 \tag{23}$$

[1] These equations are often said to be obtained by making the coefficients of the "resonance terms" equal to zero. Let us justify the term and at the same time give another, more convincing solution of these equations. Namely, instead of (21), consider

$$\ddot{x} + \omega^2 x = F(\phi, \dot{\phi}) + (\omega^2 - \omega_0^2)x - 2h\dot{\phi} = f_1(\phi, \dot{\phi}).$$

Assuming the existence of oscillations, close to a sinusoidal oscillation $x = A \cos \omega t$, we will be dealing with forced oscillations of a harmonic oscillator *without friction:*

$$\ddot{x} + \omega^2 x = P(A) \cos \omega t + (Q(A) + 2h\omega A) \sin \omega t + G(A,t).$$

As we know, the oscillations will fail to grow indefinitely only if the coefficients of the resonance terms of the external force are equal to zero, and this yields again (23).

which determine the A and ω of the approximately *sinusoidal self-oscillations.*

Even under our assumptions the amplitude and frequency given by (23) are generally not the exact amplitude and frequency of the fundamental tone since in passing to the "forced" problem we replaced the precise solution by $A \cos \omega t$. Replacing our zero approximation by the "first" approximation[1]

$$x_1(t) = A \cos \omega t + z_1(A,t),$$

we get, instead of (23), some other conditions on A and ω. Then taking further approximations we obtain a sequence of values of A and ω whose convergence to a precise solution requires a special investigation. We shall discuss this when we take up Poincaré's quantitative methods.

Many investigators, notably Möller and van der Pol, assume that self-oscillations are approximately sinusoidal. The precise methods of Poincaré apply conveniently under this assumption. In many cases, however, as in the various multivibrators which are widely used in measuring devices, this is not a satisfactory assumption. We shall pay special attention to these cases later.

Self-oscillating systems in which one of the oscillating parameters (spring constant, capacitance or inductance) is of secondary importance compared with friction will be called *relaxational,*[2] in contrast with systems of Thomson type in which the primary role is played by capacitance and inductance.

Frequently one may simply neglect the unimportant elements, notably the mass or the inductance. This brings about a degeneracy of the basic differential equation and the system may then be viewed as producing oscillations with discontinuities, with the related conditions of jump. These questions are dealt with in the next chapter.

[1] Let us note that if the "first approximation" represents with sufficient precision the desired periodic motion which, according to our assumption, should be close to a sinusoidal motion, then the harmonic coefficient should be small. Even when this holds, we cannot say whether A and ω obtained from equation (23) will represent the fundamental tone of our solution with sufficient precision, and what will be the harmonic coefficient for further "approximations."

[2] This type of oscillation was first obtained as a periodic discharge of a condenser through a resistance. The term "relaxational" was borrowed from mechanics. In mechanics "relaxation," a gradual disappearance of elastic deformation in a medium possessing friction, is analogous to the discharge of a condenser through resistance.

CHAPTER IV

Dynamical Systems Described by a Single Differential Equation of the First Order

§1. INTRODUCTION

We have already seen, for instance, in the rectilinear motion of a material point attracted by a spring, that for a very small mass m or small spring constant k, the basic differential equation degenerates from the second to the first order. Similarly in an L,R,C circuit with L small or C large. Whenever in addition one of the elements is non-linear, even discontinuous, there arise non-linear problems and possible non-linear oscillations. As an example, instead of the complete equation for a self-oscillatory system

$$m\ddot{x} + kx = \phi(\dot{x}) \tag{1}$$

we have, when the mass m (or the inductance in an electrical system) is negligible, the degenerate equation

$$kx = \phi(\dot{x}). \tag{2}$$

This may sometimes be solved for $\dot{x}$ yielding an equation

$$\dot{x} = f(x) \tag{3}$$

and it is mainly with equations of these two types that we shall deal in this chapter.

It may be noted that, when the spring constant k is negligible (or the capacitance very great in an electrical system), (1) reduces to

$$m\ddot{x} = \phi(\dot{x})$$

which, upon setting $\dot{x} = y$, reduces to

$$m\dot{y} = \phi(y)$$

which is again of type (3).

Regarding (2) or (3) several important observations may be made in connection with oscillatory systems. The first is as follows. If we set $\dot{x} = y$, then both yield the same graph in the phase plane x,y:

$$kx = \phi(y), \tag{4}$$

$$y = f(x). \tag{5}$$

If there is to be an oscillation, the graph must represent a closed curve; hence some horizontal and some vertical must meet the curve in more than one point. This means that neither $\phi(y)$ nor $f(x)$ can be single-valued without exception. It excludes, for instance, the possibility that $f(x)$ be analytical *without exception* for all values of x since this would imply that $f(x)$ is single-valued. Thus in the case of periodic motion one will have to expect functions only analytical in part, possible presence of radicals, or even discontinuities.

The second observation refers precisely to the possible discontinuities or jumps. In this connection one must bear in mind the general

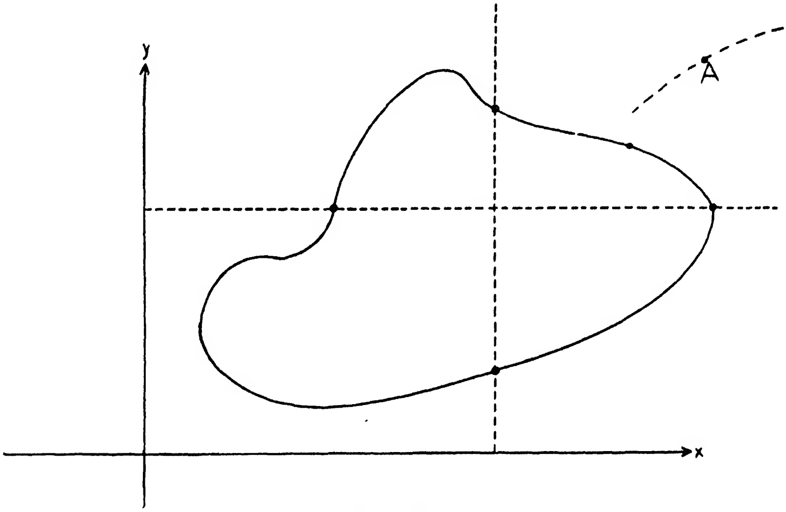


FIG. 121.

postulate of *no change in the total energy stored in the system*, which has also been referred to as the *condition of the jump*. It is only necessary to retain here the following consequences which alone matter and may be justified on sound physical grounds:

(a) In a mechanical system one may admit abrupt changes of the velocity $\dot{x}$ but the displacement x must be continuous.

(b) In an electrical system with a large inductance L the current i through L must be continuous, while if there is a large capacitance C the potential difference V across C must be continuous.

Our last observation is this: In our degenerate systems we have only one path represented by (4) or (5). Hence we cannot have arbitrary initial conditions and expect to have an oscillation start forthwith. If the initial position A is not on the unique closed trajectory, the motion will "do something" before the oscillation

starts. What actually happens is that at the beginning one must take account of the neglected elements (mass, inductance, etc.) and go back to the basic equation (1) whose paths cover the plane. One will start from A, and somewhere when $m\ddot{x}$ becomes very small, together with a jump condition, one will go over to (2) or (3) and the oscillation will take place. Usually this transient initial stage may be dismissed as of no practical importance.

Returning to our general theory, equations (2) and (3) call merely for a phase line, the $x(t)$ line, and this will be considered first. At the same time it will be convenient to use the time-space diagram or plane (t,x). Later when we come to deal with oscillations proper, the phase plane (x,y) will again be utilized. In electrical phenomena it may well be a (v,i)-plane or something similar, but "phase plane" and related mechanical terminology will be preserved for convenience.

§2. Existence and Uniqueness Theory

Let us examine the (t,x)-plane. The solution of our equation, $x = \phi(t)$, is represented by curves in the plane. We shall call these curves integral curves (they should not be confused with the paths in a phase plane).

Let us assume as initial conditions $x = x_0$ when $t = t_0$: we are given a point (t_0,x_0) in the plane. We ask then under what conditions does there exist a solution of our differential equation satisfying these initial conditions, i.e. when does there exist an integral curve passing through the point (t_0,x_0). The answer is given by Cauchy's theorem relative to the existence of a unique solution of a differential equation. It asserts here that, if in the interval from $x_0 - a$ to $x_0 + a$ the function $f(x)$ is an analytic function, then it is possible to find numbers a,b $(b > 0)$ such that in the interval from $t_0 - b$ to $t_0 + b$ there exists a unique solution $x(t)$ such that $x(t_0) = x_0$. This solution is an analytic function of t in the interval $t_0 - b, t_0 + b$. Weaker conditions than analyticity may be considered as the reader will see by reference to standard texts.

Geometrically speaking, the interval $x_0 - a, x_0 + a$ cuts out on the (t,x)-plane a strip parallel to the t-axis. The interval $t_0 - b, t_0 + b$ determines on this strip a quadrilateral containing the point t_0,x_0. The Cauchy theorem asserts that in the interval $t_0 - b, t_0 + b$ there passes a unique integral curve through the point (t_0,x_0). Naturally the question arises whether this solution can be extended, i.e. whether there exists a solution of our equation in an interval greater than $t_0 - b, t_0 + b$, coinciding with the solution corresponding to the interval

$t_0 - b, t_0 + b$. To answer this question let us examine the point $(t_0 - b, x_1)$ representing the limit of the points of our integral curve when $t \to t_0 - b$. Here two cases may arise: if x_1 is a point of analyticity of the function $f(x)$, then we can apply to this point the same theorem and obtain a solution on a larger interval; if x_1 is a singular point, then Cauchy's theorem is not applicable and the unique extension may not be possible.

It is clear that the same reasoning may be applied to the other end of our portion of the integral curve—to the point $(t_0 + b, x_2)$.

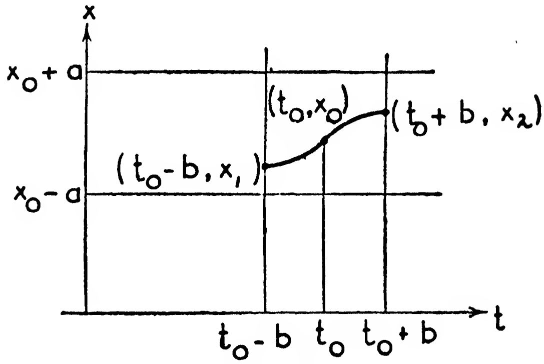


FIG. 122.

Thus we see that our integral curve will be extended, in any case until x reaches a value where $f(x)$ ceases to be analytic. If the function $f(x)$ is analytic along the total straight line, the solution will extend until x goes to infinity. If x does not go to infinity, the solution will be valid from $t = -\infty$ to $t = +\infty$.

Even if there exists a singular point, cases where the solution is valid from $t = -\infty$ to $t = +\infty$ are possible. In these cases the solution runs, for example, between two straight lines parallel to the t-axis whose ordinates are singular points of the function $f(x)$.

To sum up: the whole (t,x) plane can be divided into strips parallel to the t-axis, whose ordinates are singular points of the function $f(x)$. Within each strip there passes through each point a unique integral curve. These curves are analytic curves and do not cross one another inside the strip.

Across the boundaries of the strips various things may occur. The boundaries may be crossed continuously, or a discontinuity may arise. Let us examine an example having physical interest where the Cauchy conditions are not fulfilled: a falling body of mass m with zero initial velocity. According to the law of conservation of energy we have:

$$\frac{mv^2}{2} = mg(x - x_0).$$

Taking the positive root (we are limiting our investigation to motions in one direction), we obtain:

(6) $$\dot{x} = +\sqrt{2g(x - x_0)}.$$

Let us find a solution of this equation corresponding to the initial

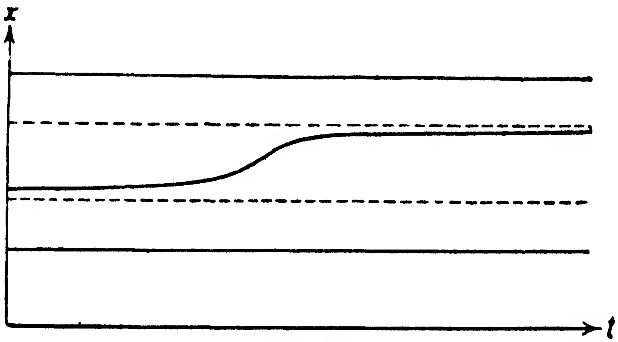


FIG. 123.

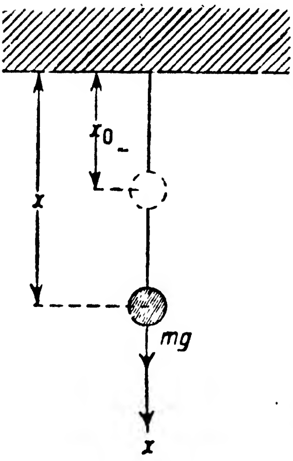


FIG. 124.

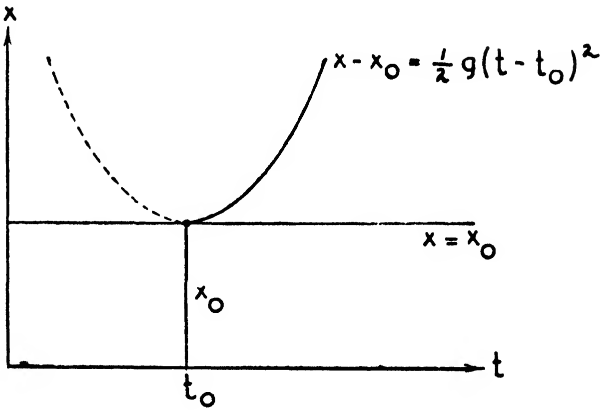


FIG. 125.

conditions $t = t_0$ and $x = x_0$. It is easy to see that for this value of x the function $f(x) = \sqrt{2g(x - x_0)}$ is not holomorphic since the derivative $f'(x)$ becomes infinite when $x = x_0$, and consequently $f(x)$ cannot be expanded in Taylor series at this point. Thus on the (t,x) plane along the straight line $x = x_0$ the conditions of the Cauchy theorem are not satisfied. What happens here may, of course, be settled by

direct integration, and we find that (6) has for the given initial conditions the solution, representing a parabola:

$$x - x_0 = \tfrac{1}{2}g(t - t_0)^2. \tag{7}$$

In these parabolas we have to consider only the branch to the right of the axis of symmetry since the radical has been taken positive and so $\dot{x} > 0$.

Besides this solution our equation has another solution $x = x_0$ satisfying the same initial conditions. It may be obtained by the ordinary rules for the envelope of the family of parabolas (7) with a variable parameter t_0. Thus we can see (Fig. 125) that through every point of the straight line $x = x_0$ there pass two, instead of one, integral curves, i.e. the solution is not unique. It is easy to indicate the physical meaning of this multiple solution. Our investigation of the falling body is based on the law of conservation of energy and not on Newton's law. From the point of view of the law of conservation of energy the body can, in accordance with the initial conditions, either fall with uniformly accelerated velocity or else remain at rest. This illustrates once more the well-known circumstance that the law of conservation of energy is not sufficient to establish the law of motion even in the case of systems with one degree of freedom.

§3. Dependence of the Solution upon the Initial Conditions

Since in real systems the initial conditions are never known exactly one must know what happens to the solution of the differential equa-

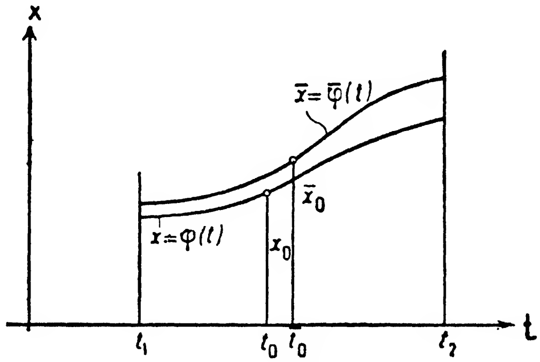


Fig. 126.

tion (3) when the initial conditions (x_0, t_0) undergo small changes. The answer is given by the theorem asserting that the solution varies continuously with the initial conditions. As we have seen, the initial conditions determine the integral curve wherever the conditions of

Cauchy's theorem are fulfilled. Thus, when they are given, x is a function of time only, so that we can then write $x = \phi(t)$. The solution, however, does not depend on t alone but is also a function of the initial conditions t_0,x_0, and therefore we must write $x = \phi(t,x_0,t_0)$ or rather $x = \phi(t - t_0,x_0)$, as the system is autonomous and so x depends obviously only on $t - t_0$. Let us assume that the integral curve $x = \phi(t)$ passes through the point (t_0,x_0) and does not leave the analytic region of $f(x)$ in the interval $t_1 < t < t_2$, which we assume includes the point t_0. The continuity theorem asserts that, if we change the initial conditions slightly, we obtain in the interval of time from t_1 to t_2 a curve $x = \bar{\phi}(t)$ close to the initial curve. Let us formulate this theorem precisely: during a given interval of time from t_1 to t_2 where the solution $x = \phi(t)$ does not leave the analytic region, there exists for arbitrary small ϵ ($\epsilon > 0$) a positive number δ such that

$$|\phi(t) - \bar{\phi}(t)| < \epsilon, \qquad (t_1 < t < t_2),$$

whenever

$$|x_0 - \bar{x}_0| < \delta, \qquad |t_0 - \bar{t}_0| < \delta, \qquad \{x_0 = \phi(t_0);\, \bar{x}_0 = \bar{\phi}(\bar{t}_0)\}.$$

§4. VARIATION OF THE QUALITATIVE CHARACTER OF THE CURVES IN THE (t,x)-PLANE WITH THE FORM OF THE FUNCTION $f(x)$

Let us assume that in the basic equation (3) $f(x)$ is an analytic function for all values of x, and that $f(x) = 0$ has no real roots. Then $\dot{x}$ preserves the same sign, say $+$, for all t and all the solutions of (3) are monotonic increasing functions when t increases from $t = -\infty$ to $t = +\infty$. On the other hand, if $f(x) = 0$ has real roots $x_1,x_2, \cdots ,x_n$, they will obviously correspond to equilibrium states. The corresponding integral curves in the (t,x)-plane are straight lines parallel to the t-axis and decompose the plane into horizontal strips. Since integral curves cannot cross, each is wholly within one of the strips and consequently is monotonic since the sign of $f(x)$ is fixed within a strip. Furthermore, it is easy to see that, if the integral curve is in the strip bounded by the horizontal lines $x = x_i$, $x = x_{i+1}$, it will approach asymptotically one of these straight lines when $t \to +\infty$ and the other when $t \to -\infty$. If the integral curve is in the part of the plane limited by a single horizontal line, the integral curve goes to infinity for some finite t or when $t \to +\infty$ or $t \to -\infty$; on the other side it tends to the limiting straight line.

Thus given $f(x)$ it is easy to determine the qualitative character of the curves in the (t,x)-plane.

Observe once more that when $f(x)$ is analytic the curves cannot be periodic since they are monotonic. This remark will be useful later.

§5. Motion on the Phase Line

We continue to consider the general system (3) where f is analytic and non-zero except possibly at a finite number of points $x_1, \cdots, x_n$. Since a motion is completely determined once x is known as a function of t, a phase line, the x-axis, representing the displacement, is all that is required. At each point P other than one of the points x_i the velocity has a definite sign and so the intervals $x_i < x < x_{i+1}$ or the half-lines

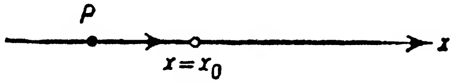


Fig. 127.

$x < x_1$, $x_n < x$, are described in a definite way and regardless of the initial position. For instance, if $f = x^2 - 1$, so that the equation is $\dot{x} = x^2 - 1$, the points to be marked are $x_1 = -1$, $x_2 = +1$. The function f is negative between these points and positive outside. Hence to the right of $+1$ or to the left of -1 the motion is always forward, while between -1 and $+1$ it is always backward.

It may be observed that, according to the case, an end-point x_i may be approached in a finite or an infinite time; no rule may be given for this behavior.

Let two representative points P_1,P_2 correspond to solutions $x_1(t),x_2(t)$, both of which start at time t_0 within the same interval $x_i < x < x_{i+1}$ of application of Cauchy's theorem. Let us follow their motion during a certain finite interval of time T during which P_1 does not leave the analytic region. The theorem on the continuity of the dependence of the solution on the initial conditions states that it is always possible to find a positive δ depending on the values of T and of ϵ, both positive, such that $|x_1(t) - x_2(t)| < \epsilon$ for $t_0 \leqslant t \leqslant t_0 + T$ if $|x_1(t_0) - x_2(t_0)| < \delta$.

Given the graph $z = f(x)$, one can divide the phase line into trajectories (segments or half-lines) and indicate the direction of the motion of the representative point along the trajectories. Figure 128 shows an example of such construction. It is clear that, if $f(x)$ is analytic throughout, the fundamental elements defining entirely the character of motions on the phase line are the zeros of $f(x)$, and they are in fact the states of equilibrium. Given the equilibrium states and their stability, it is possible to draw a qualitative picture of pos-

sible motions. In particular, it can be seen at once that, when $f(x)$ is an analytic function along the whole straight line, periodic motions are impossible.

The general behavior of the integral curves in the (t,x)-plane can be described if the character of motions of the representative point on

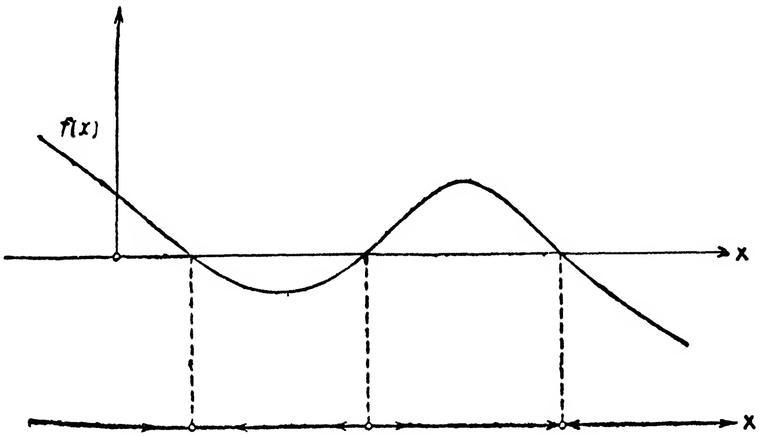


FIG. 128.

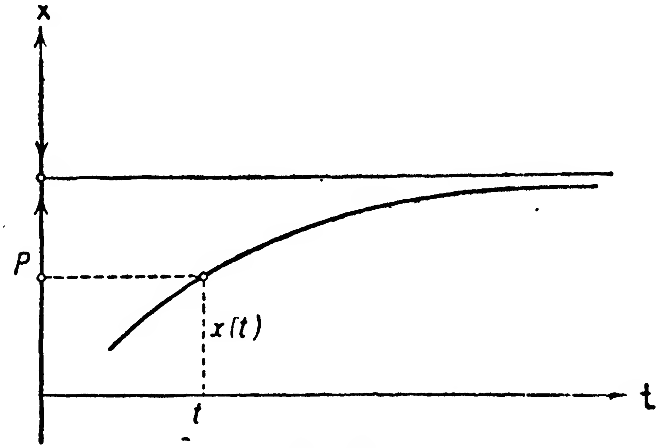


FIG. 129.

the phase line is known. Let the line coincide with the x-axis. We assume that the representative point moves along the phase line, and construct in the plane a point whose abscissa is t and whose ordinate is equal to the displacement of the representative point along the x-axis at the time t. The abscissa of this point is the time and therefore it is variable. The ordinate, generally speaking, varies also because the representative point moves. Consequently, our point will

move in the plane describing a certain curve. This curve is the integral curve of our equation.

§6. Stability of States of Equilibrium

We have previously given a definition of stability of equilibrium according to Liapounoff. We have stability according to Liapounoff whenever for any small positive ϵ and time t_0 there exists a positive δ such that, if $|x(t_0) - x_0| < \delta$, then $|x(t) - x_0| < \epsilon$ for $t \geqslant t_0$.

For our system (3) and $\dot{x}$ analytical about x_0, Liapounoff gives a definite method for answering the question. We have then $f(x_0) = 0$ and are primarily interested in small values of $\xi = x - x_0$. Substituting $x = x_0 + \xi$ and expanding $f(x)$ in Taylor series, we find

$$\begin{aligned}\dot{\xi} &= f(x_0) + \frac{(x - x_0)}{1!} f'(x_0) + \frac{(x - x_0)^2}{2!} f''(x_0) + \cdots \\ &= f'(x_0)\xi + f''(x_0)\frac{\xi^2}{2} + \cdots\end{aligned}$$

which we write

$$\dot{\xi} = a_1\xi + a_2\xi^2 + \cdots, \tag{8}$$

$$a_1 = f'(x_0), \qquad a_2 = \frac{f''(x_0)}{2}, \qquad \cdots.$$

Liapounoff's method consists in replacing (8) by the linear part

$$\dot{\xi} = a_1\xi \tag{9}$$

called the equation of the first approximation. The integral of (9) is well known to be

$$\xi = ce^{\lambda t}, \qquad \lambda = a_1 = f'(x_0).$$

Liapounoff affirms that, if $\lambda < 0$, the equilibrium state is stable; if $\lambda > 0$, the equilibrium is unstable; if $\lambda = 0$, the equation of the first approximation is inadequate for determining stability. Thus Liapounoff affirms that in certain cases the equation obtained by neglecting the non-linear terms can solve the question of stability of non-linear equations.

In the simple case under consideration it is very easy to justify the validity of the method. Multiplying both sides of (8) by ξ we obtain:

$$\frac{1}{2}\frac{d(\xi^2)}{dt} = F(\xi) = a_1\xi^2 + a_2\xi^3 + \cdots = a_1\xi^2\left(1 + \frac{a_2}{a_1}\xi + \cdots\right).$$

For ξ very small the parenthesis is near unity and hence $F(\xi)$ near $a_1\xi^2$, i.e. it has then the sign of $a_1 = f'(x_0)$. Thus if $f'(x_0) < 0$, ξ^2 decreases near zero and for $f'(x_0) > 0$ it increases. In other words, when x is slightly deviated from x_0, it tends to return to it in the

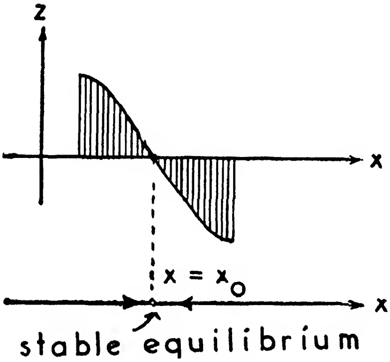


FIG. 130.

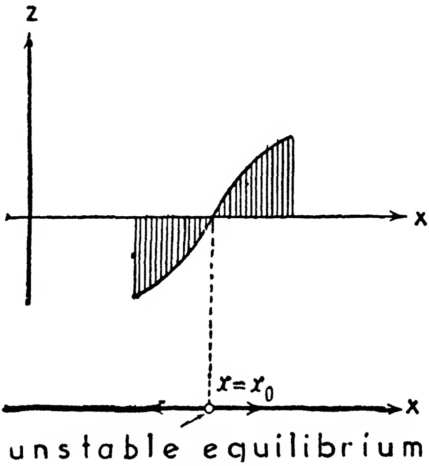


FIG. 131.

first case and to draw away from it in the second. That is to say, x_0 is stable for $f'(x_0) < 0$, unstable for $f'(x_0) > 0$.

One may investigate stability directly from the properties of the graph $z = f(x)$ near the point $(0,x_0)$. There are three possibilities illustrated by Figs. 130, 131, 132 which really tell their own story. Thus in the case of Fig. 130 the graph has negative slope at the point

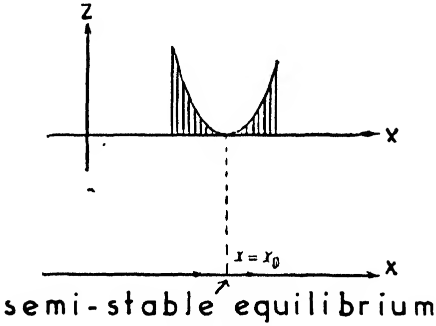


FIG. 132.

x_0 and so we have $f'(x_0) < 0$ or stability. The second, Fig. 131, corresponds to instability, while Fig. 132 corresponds to stability to the left of x_0 and instability to the right. We have here what may be termed *semi-stable* equilibrium.

§7. Dependence of a Motion upon a Parameter

In physical systems the nature of the motion is apt to depend upon multiple factors whose small changes may materially affect it. To idealize this situation one may conveniently assume that the system depends upon a single parameter λ, and discuss the variation of the solution of

$$\dot{x} = f(x,\lambda)$$

when λ varies.

Let us assume f analytic in both x and λ for all values of x and λ. As we have seen, the motion is completely determined by the states

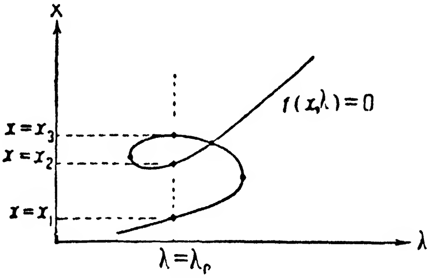


Fig. 133.

of equilibrium on the x line. They are the solutions in x of

$$f(x,\lambda) = 0. \tag{10}$$

On the (x,λ)-plane, (10) defines a certain curve. To every value of λ there corresponds one or several states $x_1,x_2, \cdots ,x_n$ whose coordinates are functions of λ. In accordance with our previous remarks an equilibrium state is stable if

$$f_x(x,\lambda) < 0$$

and unstable if

$$f_x(x,\lambda) > 0.$$

Thus we can see that the theory of the dependence of a dynamic system defined by one equation of first order on the parameter is a *precise copy* of the theory of the dependence of the equilibrium states of the simplest conservative system with one degree of freedom on a parameter. There will be branch points, changes of stability, etc. This will be best brought out by the examples to follow.

1. Voltaic arc in an R,L series circuit. As we have said before, the investigation of mechanical or electrical systems leads to one equation of first order if we neglect one of the oscillating parameters.

As an example of such a system we may take a voltaic arc connected to a battery through a resistance R and inductance L. This scheme would lead to an equation of second order if we took into account, for example, the stray capacitance of the coil. If we neglect this stray capacitance as well as all the other parasitic parameters, we arrive at one non-linear differential equation of the first order. The non-linearity is due to the fact that the arc as a conductor does not follow Ohm's law, i.e. the current through the arc is a non-linear function of the applied voltage. The relation between the voltage and the current is given graphically by the so-called *arc characteristic* $v = \phi(x)$ or $x = \psi(v)$ where v is the voltage and x the current.

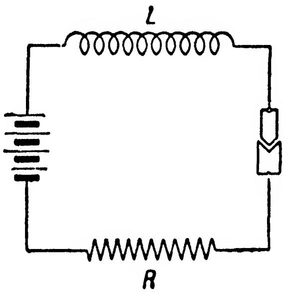


FIG. 134.

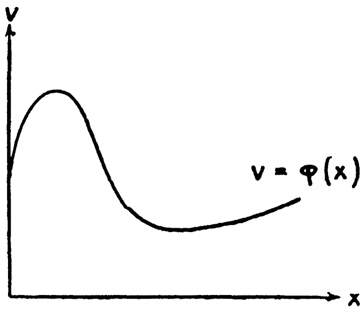


FIG. 135.

The differential equation relating the voltage and the current is

$$L\dot{x} + Rx + \phi(x) = E$$

or

$$\dot{x} = f(x) = \frac{E - Rx - \phi(x)}{L}. \tag{11}$$

The states of equilibrium are given by

$$f(x) = E - Rx - \phi(x) = 0.$$

In order to find the roots of this equation one usually constructs two auxiliary curves: the straight line $v = E - Rx$ and the characteristic $v = \phi(x)$, and finds their intersection points. We shall also use these auxiliary curves in order to find their difference (the lower curve in Fig. 136), which is proportional to the right-hand part of (11), i.e. it represents on a certain scale the function $f(x)$. Given $f(x)$ we can at once construct trajectories on the phase line (Fig. 137). In the case under investigation there exist three equilibrium states, x_1,x_2,x_3. According to our previous considerations the first and the last are stable while the middle one is unstable.

We shall now discuss what happens when one varies the parameters E,R. Consider first E as a variable parameter. According to the general rules we construct on the (x,E)-plane the curve $f(x,E) = 0$ or $E - Rx - \phi(x) = 0$. This curve, as we can see from the graph, has two vertical tangents $E = E_1$ and $E = E_2$ and hence there are two branch points E_1 and E_2. The value E_2 corresponds to such a large

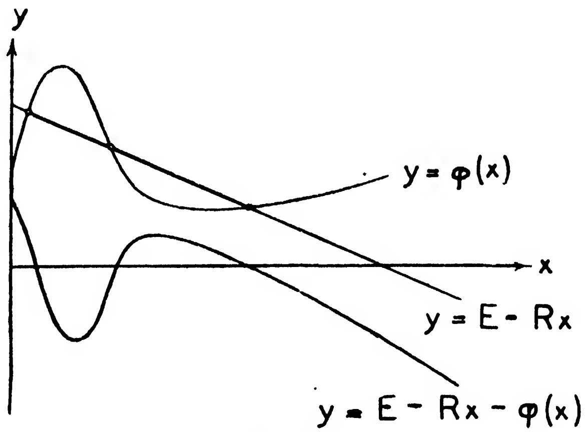


FIG. 136.

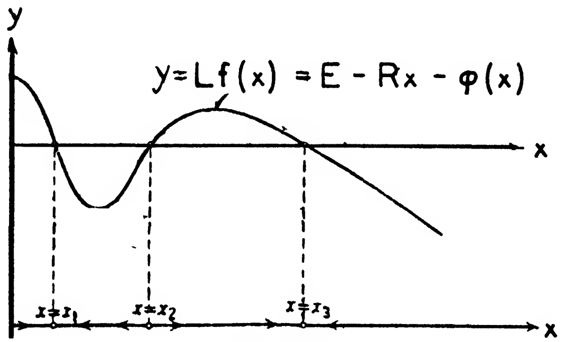


FIG. 137.

voltage on the battery (for a given R) that the equilibrium states x_1 and x_2 coincide and disappear, so that, when E increases further, x_3 is the only stable state of equilibrium left and it corresponds to a sizable current. The value E_1 corresponds to such a small voltage on the battery (for a given R) that the equilibrium states x_2 and x_3 coincide and disappear; when E decreases further, there is left only one stable equilibrium state, namely x_1, and it corresponds to an insignificant current. The diagram shows that, if E changes slowly and continuously, we shall have abrupt changes of the current of the arc circuit at

the branch points. The current will increase abruptly from x_6 to x_7 and drop abruptly from x_4 to x_5. The graph of the stationary current against the voltage of the battery has a hysteresis character. In an analogous way one can construct a diagram for given E and R variable.

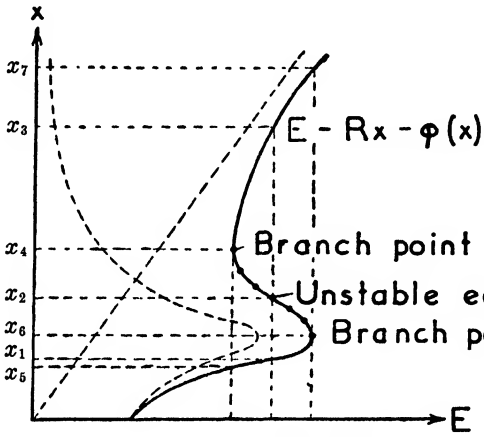


FIG. 138.

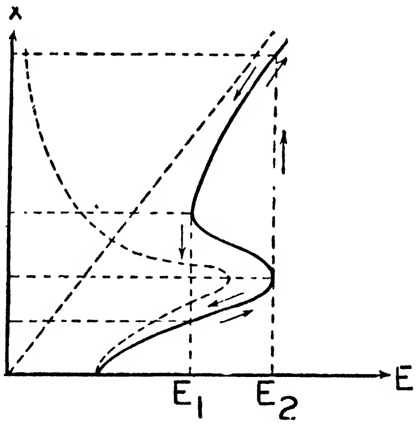


FIG. 139.

2. Single phase induction motor. As a second example let us examine a single phase induction motor. Owing to certain defects of motors of this type to be mentioned, they are useful only for low power and low starting loads (for example, for small fans).

The graphs of the torque $M(\omega)$ and of the frictional moment $m(\omega)$ (bearings plus air resistance to rotor) in terms of the angular velocity ω are indicated in Fig. 140. If I is the moment of inertia of the rotor, its equation of motion is

$$I\dot{\omega} = M(\omega) - m(\omega).$$

The equilibrium states are given by

$$M(\omega) - m(\omega) = 0.$$

In order to find the roots of this equation one constructs two auxiliary curves $z = M(\omega)$ and $z = m(\omega)$ and one finds their intersection.

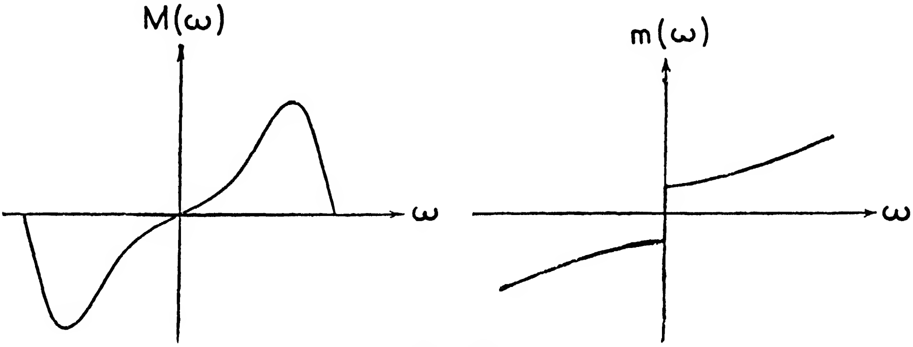


FIG. 140.

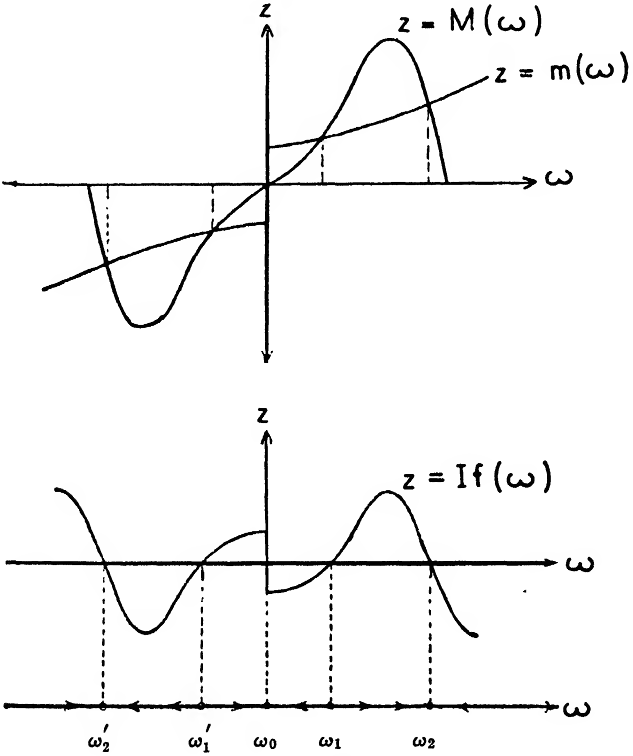


FIG. 141.

Proceeding as in the previous example we find the function $f(\omega)$ and we construct the trajectory on the phase line. In our case we have three stable equilibrium states $\omega = \omega_0$, $\omega = \omega_2$, $\omega = \omega_2'$ and two

unstable equilibrium states $\omega = \omega_1$ and $\omega = \omega_1'$. The variation of the situation with m and M may be dealt with as in the previous example.

The stability of the equilibrium $\omega = \omega_0 = 0$, corresponding to complete rest, shows that the rotor does not start by itself but has to be in some way thrown out beyond the state ω_1 or ω_1', i.e. one must impress upon it a rotation at least ω_1 in one or the other direction after which it reaches itself the normal number of revolutions corresponding to ω_2. The motor can rotate in both directions (two equilibrium states ω_2 and ω_2') and the direction of the established rotation depends only on the direction into which the rotor is "thrown." In order to eliminate the necessity of starting the rotor, special devices are used (auxiliary coils, poles, etc.) which cause an asymmetry of the picture of rotation in the two directions. This produces a certain initial torque and the motor starts by itself, of course always in the same direction. The asymmetry does not make the initial rotational moment sufficiently large, for which reason the starting of the motor is never very smooth. This greatly restricts the range of application of these motors.

§8. Relaxation Oscillations, Mechanical Examples

The term *relaxation oscillations* is generally applied wherever the oscillations have a more or less zigzag or sawtooth aspect. It is precisely the situation which we shall find here.

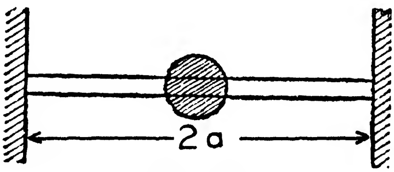


FIG. 142.

In a system governed by the basic differential equation (3) no oscillation can occur if $f(x)$ is analytic throughout. Furthermore, when $f(x)$ is single-valued, oscillations can only occur when the equation fails to describe the system somewhere. This may be due to physical reasons or else because f ceases to be analytic. In either case one must have recourse to physical considerations such as the jump condition to continue the motion beyond the doubtful point.

As an example of the first case, physical reasons, take an elastic ball moving without friction on tracks between two parallel solid walls at a distance $2a$ apart. The equation of motion of the ball is $\dot{y} = 0$; therefore $y = v_0$ (y = velocity of the ball). It is obvious in advance that this equation ceases to be valid when the ball reaches a wall. In

order to determine the behavior of the ball beyond we introduce the additional assumption that the ball rebounds according to the law of elastic collision. Its motion after this collision is defined by the same equation but with initial velocity $y = -v_0$. It is easy to see that the ball will perform oscillations between the two walls. The character of the motion in the phase space, accompanied by abrupt changes

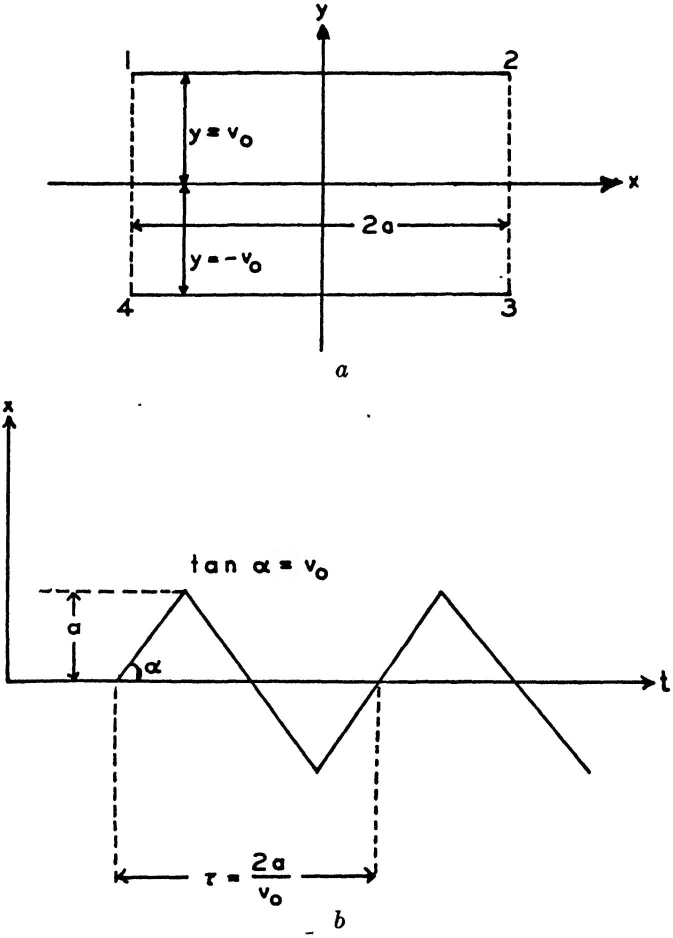


Fig. 143.

representing the collisions, is shown in Fig. 143*a*. The form of oscillations is shown in Fig. 143*b*.

As to the second possibility, that $f(x)$ ceases to be analytic, we are mainly interested in the case where it becomes infinite. Our early theory fails here and to determine what happens it is necessary to introduce new postulates suggested by physical considerations, for

example, the condition of jump. We shall examine later a series of examples bearing on this case.

The preceding example, altogether elementary, could be settled by means of an equation of type (3). In more complicated cases it is advisable to recall the initial type

$$m\ddot{x} + kx = -F(\dot{x}) \tag{12}$$

from which (3) arose. It will be remembered that we assumed $m\ddot{x}$ very small, thus reducing (12) to the type

$$kx = -F(\dot{x}); \tag{13}$$

then one solved (13) for $\dot{x}$, yielding (3). These assumptions must be kept in mind throughout the discussion.

If the mass m is small and the friction is great, the character of the motion will be as follows: when the frictional forces of the system vary within very large limits, they appreciably disturb in certain regions the relation $m\ddot{x}$ = spring force (or in general any regenerating force) so that the whole process can be definitely divided into two substantially different phases.

1. *First phase.* The spring force kx is much greater than the inertial force $m\ddot{x}$, and consequently the motion is governed mostly by the spring force and the frictional force. During this phase x changes substantially and the change in the velocity $\dot{x}$ is relatively slow, i.e. the acceleration is not large. Owing to the smallness of the acceleration and the mass, their product does not play an important role in the equation of motion. However, as x varies considerably, the spring force likewise undergoes an appreciable change.

2. *Second phase.* The inertial force $m\ddot{x}$ is much greater than the spring force, and consequently the motion is mainly governed by the inertial and frictional forces. Since the mass is small, the acceleration is very great. During this phase x does not have time to change appreciably but the velocity of the system varies sharply. Since the velocities are always limited and the accelerations are very large, this phase lasts a very short time as compared with the first.

Needless to say, when friction is small and the oscillations are nearly sinusoidal, the division into two phases is impossible since both the inertial and the spring force are all the time almost equal; when one is small, so is the other.

Let us assume that the mass of the system is extremely small so that, during the first phase where accelerations are small, it does not play any role. The frictional force $F(\dot{x})$ is a certain function of veloc-

ity, and during this whole phase the motion is governed by

$$kx = -F(\dot{x}).$$

This equation cannot, however, cover the first phase alone. If we differentiate with respect to time, we obtain:

$$F'(\dot{x})\ddot{x} = -k\dot{x}.$$

If $F'(0) < 0$, i.e. the characteristic of friction is decreasing, the equilibrium state is unstable. Consequently, the system cannot remain at rest in this state. Furthermore, if in the regions where the velocity $\dot{x}$ has a finite value, $F'(\dot{x})$ tends to zero, the acceleration $\ddot{x}$ tends to infinity, i.e. the system leaves the phase where $m\ddot{x}$ can be neglected. Of course, according to our idealization, this result—unlimitedly large acceleration—is not contradictory as long as we neglected the mass, because in the absence of mass we may admit that the velocity changes abruptly. Thus, in our idealized scheme, the limits of the first region are defined by the state of the system where $F'(\dot{x}) = 0$ and $\ddot{x} = \infty$.

In the second phase where accelerations are large, it becomes impossible to neglect the mass. The velocity changes very rapidly during very short periods of time, while the coordinate x changes very little. Moreover, the time necessary to traverse this phase is very small compared to the time required for the first phase. Consequently, the second phase has little effect on the character of the oscillatory process and does not change appreciably its period (time is short), its "amplitude" (the variation of the coordinate is very small), and its form (the derivative changes very rapidly during a very short period of time). Thus in an oscillatory process one may disregard the second phase, and determine by some jump condition the final state which the system reaches after having passed it.

To illustrate the preceding generalities we will discuss at length a system which is in substance a much simplified Prony brake. A block of small mass m fastened to a fixed frame by a spring rests upon a uniformly revolving shaft. The oscillations which we shall find are, in fact, observed in Prony brakes in practice. Neglecting the moment of inertia of the block, its equation of motion is

$$rF[(\Omega - \dot{\phi})r] = k\phi \tag{14}$$

where k is the spring constant (couple required to rotate the block by an angle unity), Ω the angular velocity of the shaft, $\dot{\phi}$ the absolute angular velocity of the block, r the radius of the block, $F(v)$ a function

representing the dependence of the frictional force upon the relative velocity $v = r(\Omega - \dot{\phi})$. A schematical characteristic of friction is shown in Fig. 144. The relative velocity is represented along the horizontal axis, and consequently $v = 0$ corresponds to joint motion of the block and shaft, while $v = v_0 = r\Omega$ corresponds to $\dot{\phi} = 0$, i.e. to the block at rest. At the left of the vertical $v = r\Omega = v_0$ lies the region where the block moves in the same direction as the shaft and at the right the region where the block moves in the direction opposite to that of the shaft. Since a stationary motion of the block with constant velocity is ruled out, $\phi = 0$ is the only equilibrium state.

In order to investigate this characteristic from the point of view of the questions we are interested in, it is necessary to keep in mind the

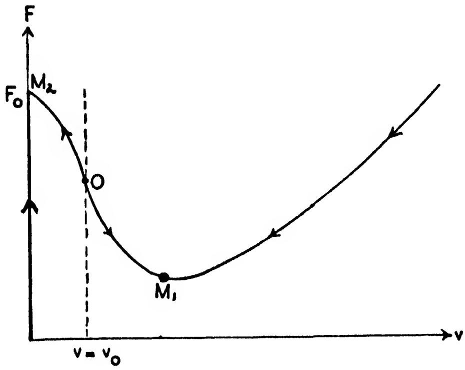


FIG. 144

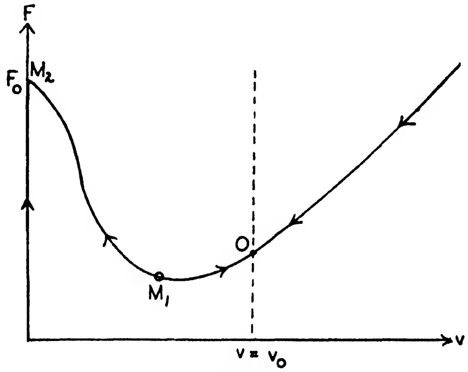


FIG. 145.

following circumstance often mentioned before: as long as the relative velocity is nil, the frictional force can have any value below the value F_0 corresponding to friction at rest. In fact, if $v = 0$ the block is at rest with respect to the shaft until the moment $k\phi = F_0 r$, i.e. when $v = 0$ any force $k\phi < F_0 r$ is balanced by the frictional force. This means that, when $v = 0$, the frictional force can have any value $F < F_0$, i.e. the characteristic of friction has a vertical branch (the heavy line in Fig. 144) coinciding with the vertical axis from $F = 0$ to F_0. The role of this vertical branch will become clear below.

From the relation (14) we infer that the graph of $F[r(\Omega - \dot{\phi})]$ against v, in Fig. 144, represents also in a suitable scale the graph of $r\phi$ against $r\Omega - r\dot{\phi}$, i.e. as measured from the vertical axis $r\Omega$ the path in the sense of §1. The direction of motion of the representative point on the curve is given by the sign of $\ddot{\phi}$. To determine it we differentiate (14) and obtain:

$$k\dot{\phi} = -r^2F'[(\Omega - \dot{\phi})r]\ddot{\phi}. \tag{15}$$

Consequently, if $F' > 0$, ϕ and $\ddot{\phi}$ are of opposite sign, while if $F' < 0$ they are of the same sign. If, furthermore, we take into account that ϕ changes its sign when passing through zero, i.e. when $v = v_0$, the direction of motion of the representative point will be as indicated by the arrows in Fig. 144. If the angular velocity of the shaft is so small that v_0 is situated on the dropping portion of the characteristic, i.e. $F'(r\Omega) < 0$ (Fig. 144), the equilibrium state $\phi = 0$ (block at rest) is unstable. If on the contrary Ω is so high that v_0 is on the ascending portion of the characteristic, $F'(r\Omega) > 0$ (Fig. 145) and the equilibrium state $\phi = 0$ is stable. In both cases, when ϕ is sufficiently large, the representative point moves in the direction of small ϕ.

From the relation

$$\ddot{\phi} = \frac{-a\phi}{F'[r(\Omega - \dot{\phi})]}, \qquad a = k/r^2 > 0,$$

we see that, as the representative point tends to the minimum or maximum position on the curve, i.e. the points where $F' = 0$, the acceleration $\to \infty$. This means that the system goes into a situation where the inertial couple $I\ddot{\phi}$ ceases to be negligible and the representation by the differential equation of the first order (14) is inadequate. This is where we will expect to apply a "condition of jump" with a discontinuity in the velocity but not in the angle ϕ itself. That is to say, the representative point will undergo a rapid horizontal displacement

From the frictional characteristic one may readily show that the block undergoes a periodic motion when $F'(r\Omega) < 0$ as in Fig. 146. At the beginning the block is caught by the shaft and moves together with it uniformly with a velocity Ω. This stretches the spring, while the frictional force increases, remaining all the time equal to the spring tension. The relative velocity is then equal to zero. The representative point in the phase plane describes the segment AB with constant phase velocity. When the moment $k\phi$ becomes equal to the moment of frictional force at rest, an abrupt change of velocity occurs, the value and the direction of the velocity change, while the elongation of the spring remains constant.

In practice the spring is a little compressed during the "jump" because the jump is not instantaneous; the smaller the moment of inertia of the block, the smaller the compression of the spring. The order of magnitude of the variation of the coordinate ϕ accompanying the jump may be calculated as follows. Since the system has a certain moment of inertia which can be computed, the variation of the kinetic

energy relative to the change in velocity can also be calculated. If we calculate the amount of kinetic energy received by the system during the velocity "jump" and assume that the system obtains all this energy from the loss of energy of the spring, we can determine the maximum possible compression of the spring accompanying the velocity "jump." In the system under investigation this compression can be made very small. Therefore, we may assume that the coordinate ϕ remains constant during the abrupt change of velocity. Consequently, the spring force and the frictional force remain constant (more precisely, almost constant) and the representative point jumps along the horizontal from B to C, which corresponds to the same value of

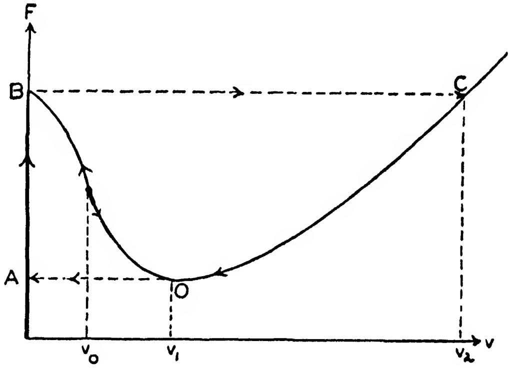


FIG. 146.

frictional force, i.e. $F(v_2) = F(0) = F_0$. Afterwards the variation of the velocity and of the coordinate is continuous; the motion is then defined by (14) and it proceeds along the arc CO in the direction of the minimum point. If the velocity of the shaft were such that $F'(r\Omega) > 0$, then $v_0 = r\Omega$ would be situated on the ascending portion of the characteristic (Fig. 145) and the equilibrium state would be stable. The representative point, after reaching it, would remain there. If Ω is such that $F'(r\Omega) < 0$ (Fig. 146), the motion reaches the point v_1 where $F = F_{\min}$ and $F'(v_1) = 0$. From this point the velocity jumps again to the point A of the vertical portion AB of the characteristic corresponding to $F_{\min}$, the coordinate remaining constant. Then the motion repeats itself; the block undergoes self-oscillations. Thus $F'(r\Omega) < 0$ is the condition of excitation of these self-oscillations. If this condition is fulfilled, the block experiences relaxation oscillations.

The amplitude of the oscillations is

$$\phi_a = \frac{r}{k}(F_0 - F_{\min}).$$

To calculate the period it is necessary to know the time required to describe the segment AB and the arc CO. As to the portions from B to C and from O to A, theoretically the motion there is instantaneous and practically it is so rapid that it does not affect appreciably the period. Since $\dot{\phi} = \Omega$ along AB, the time required to describe it is

$$T_1 = \frac{\phi_a}{\Omega} = \frac{r}{k\Omega}(F_0 - F_{\min}).$$

The time T_2 required to describe CO is obtained by integrating (15), which yields

$$T_2 = \frac{r^2}{k}\int_{\dot{\phi}_1}^{\dot{\phi}_2} \frac{F'[r(\Omega - \dot{\phi})]}{\dot{\phi}}\, d\dot{\phi}.$$

Thus when one knows the graphical representation of $F[r(\Omega - \dot{\phi})]$, one

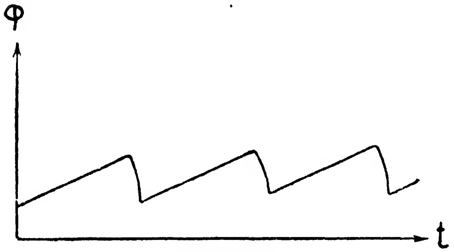


FIG. 147.

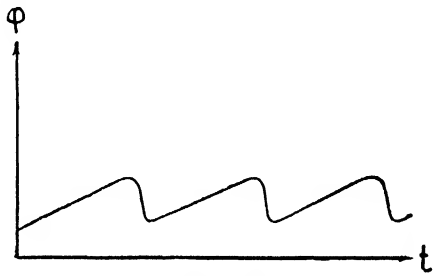


FIG. 148.

may obtain T_2 by graphical integration of the function $F'[r(\Omega - \dot{\phi})]/\dot{\phi}$. The period is $T = T_1 + T_2$.

As we have said before, the velocity actually does not change abruptly because the system does have a certain moment of inertia. Consequently, the "jump" of velocity is accompanied not by an unlimitedly great acceleration but by a great but limited acceleration. The product $I\ddot{\phi}$ plays a substantial role during this phase, and the equality between $k\phi$ and $rF[r(\Omega - \dot{\phi})]$ is upset since ϕ remains almost constant while $F[r(\Omega - \dot{\phi})]$ passes through all successive values. If, however, one adds to these two forces the inertial force, then all the forces acting upon the system will again balance.

The lines representing the jumps (dotted lines) in the phase plane will be curved lines. If the moment of inertia is sufficiently small and the moments of the spring force and friction sufficiently large, this curvature is practically unnoticeable. The character of the process changes substantially when the moment of inertia of the block increases. Experimental curves corresponding to the oscillations of

the block for different moments of inertia are represented by Figs. 147 and 148. When the moment of inertia increases and the spring force decreases further, one obtains oscillations more and more nearly harmonic and the "relaxation pendulum" becomes Froude's pendulum, i.e. a self-oscillatory system whose description requires two equations of the first order. The diagram in the phase plane (Fig. 149) corresponding to the curve relative to the first case shown in Fig. 147 is substantially different from the diagram of Fig. 150 corresponding to the oscillations of Fig. 148. One can see that when the moment of inertia becomes large, the velocity jumps disappear.

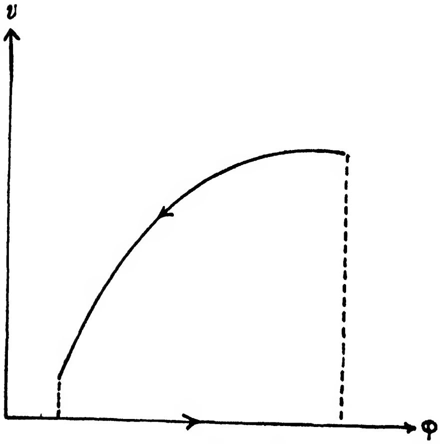


Fig. 149.

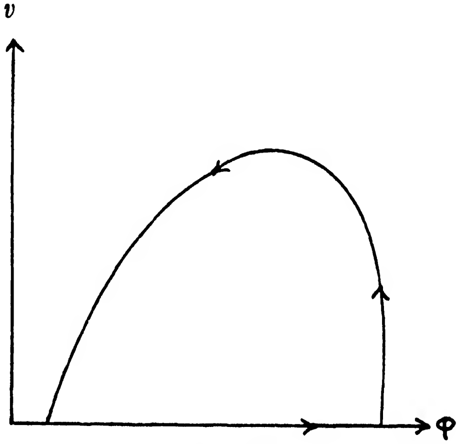


Fig. 150.

If the characteristic of friction did not rise, after the minimum, up to the value F_0, the velocity would not be able to change abruptly as the coordinate ϕ remained constant. In this case the change of the coordinate and the decrease of the spring force would take place simultaneously with the change of velocity, i.e. the representative point would move not along a horizontal line but along a curve inclined downwards, as in the case of large moments of inertia. This motion would continue until the representative point reached the ascending branch of the characteristic of friction, when (14) would become again valid and the process would continue as described before. The "frictional force" plays an important role in this transient motion, until the representative point reaches again the characteristic of friction, because the equality between the frictional force and the spring force is disrupted, giving rise to a large acceleration and an inertial force compensating the inequality between the frictional force and the spring force. In other words, if the frictional characteristic does not

ascend to the value F_0 after the minimum, our idealization cannot be applied; for if we neglect the mass of the block, we cannot determine the point C of the ascending portion of the characteristic which the representative point is to reach after the "jump."

If we neglect the mass, we are also unable to determine the manner in which the representative point reaches the phase curve if it is not situated on it at the initial moment, i.e. if the initial conditions are in conflict with the equation describing the system. The condition of jump, which in the case of linear systems settles this question, fails here. For example, if the system is situated at the initial moment

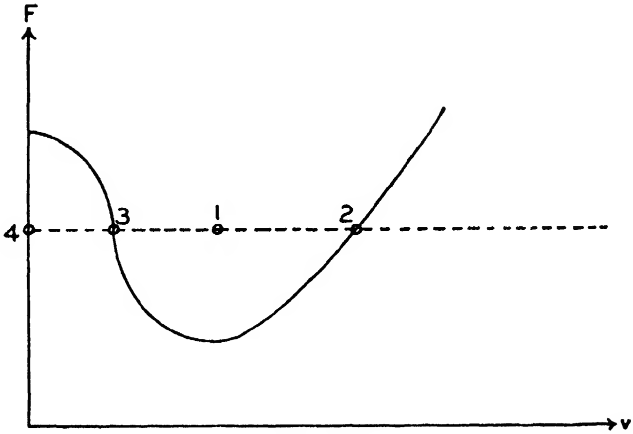


FIG. 151.

at the point 1, it can jump to the points 2, 3, or 4 and still satisfy the condition of jump. The significance of this has already been pointed out in §1.

§9. Relaxation Oscillations, Circuit with a Neon Tube

We shall now investigate the oscillations occurring under certain conditions in a circuit containing a neon tube N connected to a battery E through an ohmic resistance R, and a shunted capacitance C. In the discussion we shall neglect the stray inductance and capacitance of different elements of the circuit. As a consequence we shall have a degenerate system described by one non-linear differential equation of the first order. The non-linearity is caused by the neon tube, for which the relation between the voltage v and the current i is not determined by Ohm's law but by a more complicated non-linear relation resembling hysteresis. As far as we are concerned the most typical traits of the neon tube are the following. When the voltage is small, the current cannot pass through the tube. In order to start

the current one requires a certain firing voltage v_1. Then a current of a certain intensity i_1 is established instantaneously. When the voltage increases further, the intensity increases according to an almost linear law. When the voltage decreases from the side of large $\imath$, it may actually go below v_1; the tube remains conducting when the voltage decreases even further, the current in the tube gradually decreases, and finally it goes out instantaneously for a certain voltage v_2 and intensity i_2, where $\imath_2 < v_1$ and $i_2 < i_1$. All these traits of the neon tube are very definite; we can describe them by means of the characteristic $i = \phi(v)$ represented in Fig. 153. In fact, if we vary the voltage v on the plates of the tube in the region of small v, the tube will not fire while $v < v_1$, i.e. no current will be passing through the

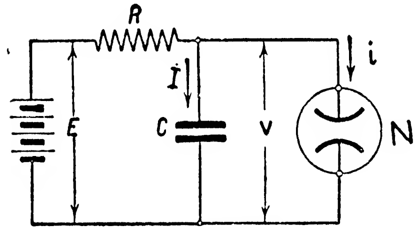


Fig. 152.

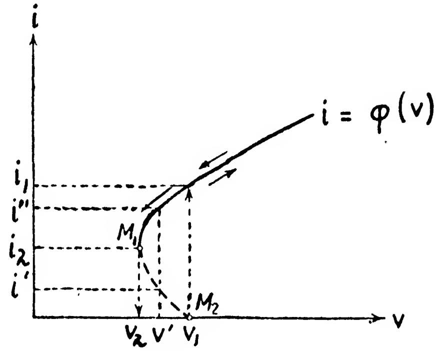


Fig. 153.

tube. The tube will fire when $v = v_1$ and a current i_1 will be established instantaneously (we are assuming that the circuit is without inductance and therefore the current can change abruptly). Conversely, when the voltage decreases, the tube goes out immediately when $v = v_2$ (since when $v < v_2$ it cannot conduct) and the intensity of the current will immediately drop to zero. Of course, every circuit has inductance and therefore the current cannot change abruptly either in the exterior circuit or in the tube itself. However if one neglects the inductance of the circuit, one may assume that the current changes abruptly, i.e. as shown by the arrows in Fig. 153.

The lower portion of the characteristic "in front" of which the representative point passes when we change the voltage on the plates of the tube does not exist in a static regime. This portion of the characteristic, however, has a quite definite significance and affects the behavior of the system in certain cases. The characteristic represented in Fig. 153 coincides, in its basic traits, with those obtained experimentally.

Let E denote the voltage of the battery, v the voltage on the plates of the capacitor, i the current in the tube, and I the current in the capacitor circuit (Fig. 152). Applying Kirchhoff's law and neglecting the internal resistance of the battery, we obtain:

$$R(i + I) + v = E, \qquad I = C\dot{v}.$$

If we eliminate I from both equations and replace i by $\phi(v)$, we obtain for v the following non-linear differential equation of the first order:

$$\dot{v} = f(v) = \frac{1}{RC}(E - v - R\phi(v)). \tag{16}$$

The equilibrium state of the system is determined by the condition

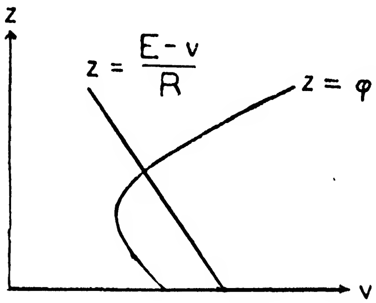


Fig. 154.

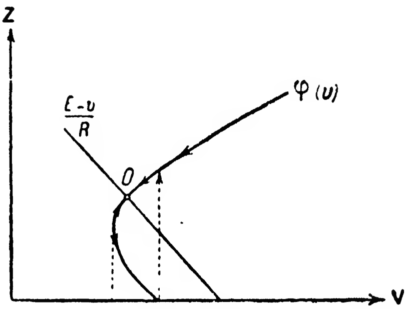


Fig. 155.

$f(v) = 0$, i.e. by the equation

$$\frac{1}{R}(E - v) = \phi(v).$$

In order to obtain the roots of this equation we construct the graphs $z = \phi(v)$ and $z = (E - v)/R$ and find their intersection. We shall limit our investigation to the case $E > v_1$, i.e. when the voltage on the battery exceeds the voltage necessary to fire the tube. Then there exists only one intersection point of the curve $z = \phi(v)$ and the straight line $z = (E - v)/R$; the position of this intersection on the characteristic depends upon the values of the parameters E and R. (See Fig. 154.) The stability of this equilibrium state is determined, as we know, by the sign of the function $f(v)$, or, what is the same, by the sign of $\dot{v}$ in the vicinity of equilibrium. It is easy to see that all the equilibrium states situated on the upper part of the characteristic are stable and those situated on the lower part are unstable. Consequently, for every value of E (where $E > v_1$) we can, by increasing R,

pass from stable to unstable equilibrium; the higher the value of E, the higher must be the critical resistance R_{cr} for which the intersection point passes onto the lower portion of the characteristic where equilibrium becomes unstable.

It is convenient to take as the path $i = \phi(v)$ itself. The direction of motion of the representative point along the curve is determined by the sign of $\dot{v}$ and is represented by arrows on Fig. 155. By means of the curve it is easy to follow the motion of the system in various cases. When R is sufficiently small and equilibrium is stable, the tube fires immediately upon being connected to the plates of the capacitor and after that the voltage on the tube and the current through it will start to decrease. The velocity of this change of voltage on the plates is determined by the parameters of the arrangement, but in any case it is always finite. The voltage on the plates will continue to drop until it reaches the equilibrium state (O in Fig. 155) in which the system will remain; the tube will merely conduct. If, however, R is so large that the equilibrium is unstable, the system cannot reach this unstable equilibrium state. As long as this unstable equilibrium is the only one available, we have sufficient ground to assume that the stationary state of the system is a periodic process.

First of all, the following question arises: Is the periodic solution compatible with the equation of the first order (16)? This equation, as we know, may admit continuous periodic solutions for v only when the function $f(v)$ or equivalently $\phi(v)$ is not single-valued. Furthermore, the voltage v on the plate of the capacitor determines the energy of the system and according to our basic postulate (§1) it cannot change abruptly. Consequently, discontinuous periodic solutions are in general impossible for v. Therefore, the necessary condition for the existence of periodic motion in the system is that the function $\phi(v)$ must not be single-valued. The characteristic of the neon tube shows that this condition is fulfilled. To every value of the argument v between v_1 and v_2 (for example, v' in Fig. 153) there correspond two values i', i'' of i. Therefore, the system may admit oscillations corresponding to a continuous variation of v. As to $\dot{i}$, i.e. the current in the circuit, it may change abruptly since the current is not associated with the energy of the system and therefore $\dot{i}$ can become infinite.

Let us determine the state in which $\dot{i} = \infty$. Since $i = \phi(v)$ we have

$$\dot{i} = \frac{d\phi}{dv}\dot{v} = \phi'(v)f(v).$$

Since the function $f(v)$ remains finite when v remains finite, $\dot{i} = \infty$ only when $\phi'(v)$ either does not exist or else ceases to be continuous, i.e. at the points M_1 and M_2 of the characteristic. At these points the current can change abruptly; this abrupt change (jump) represents the only possibility that the representative point has to leave the position into which it comes when it moves along the path from M_1 to M_2. Since an abrupt change of the current is possible ($\dot{i} = \infty$), we may assume that a jump actually takes place. The condition of jump allows us to determine the state into which the system comes. Since the voltage on the plates of the capacitor cannot change abruptly, the jumps must take place along the i-axis, i.e. as indicated by the arrows

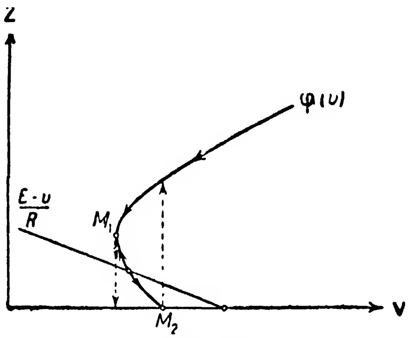


FIG. 156.

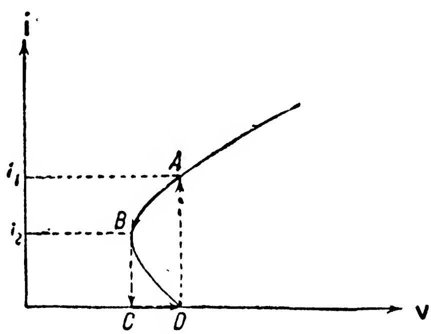


FIG. 157.

of Fig. 157. It is easy to see that the condition of jump uniquely determines the final state into which the system comes after the jump.

Thus equation (16) together with the condition of jump completely determines the behavior of the system. The motion of the representative point along the path shows that, when $E > v_1$ and the resistance R is sufficiently great, i.e. when the unique equilibrium state is unstable, the system experiences periodic motion (Fig. 157). In fact, after the tube first fires (for $E \geqslant v_1$), the current starts to drop and the velocity of this drop of current depends upon the parameters of the scheme. The representative point describes the arc AB of the characteristic (the capacitor is discharged through the tube). At the point B the current changes abruptly; it drops from i_2 to zero and the tube goes out. After that the voltage across the capacitor and on the tube will start to increase—segment CD—likewise with a finite velocity depending on the magnitude of C and R (the capacitor is charged while the tube is out). When the voltage reaches the value v_1, the tube fires again and the current increases abruptly and becomes equal to i_1. Then the process repeats itself and an oscillation is established. To the periodic

process there will correspond in the phase plane (v,i) a "closed" path consisting of two portions of the curve along which the representative point moves with a finite velocity; the extremities of these portions of the curves are "connected" by the jumps. The form of the oscilla-

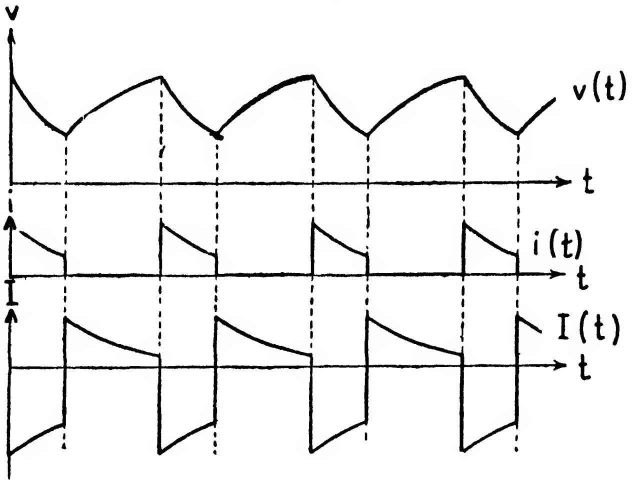


FIG. 158.

tions, i.e. the form of the curves $v(t)$, $i(t)$, and $I(t)$ is shown in Fig. 158. The forms of the portions at the beginning, before the periodic process is established, depend on the initial conditions.

As in the mechanical example, the initial conditions are not generally sufficient to determine the system (§1). Thus if the initial point is specified as A in Fig. 159, so that we have given v_0 and i_0 at the

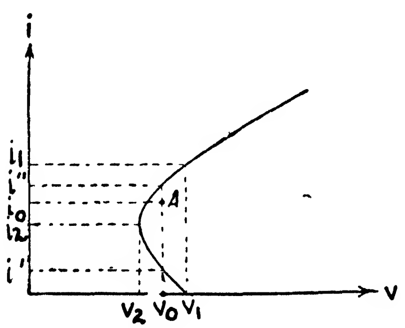


FIG. 159.

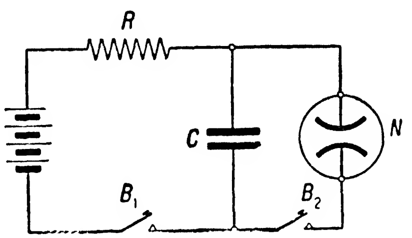


FIG. 160.

beginning, the system may conceivably jump either to i' or to i''. Here again the neglected parasitic elements will determine what happens at the outset. Our idealization is only valid when the initial conditions are such that at the beginning of the process the representa-

tive point is on the path or on the v-axis (when $i_0 = 0$). It is only then that we may determine the manner in which the periodic process is established. Thus if at the initial moment $v_0 = E$, $i_0 = 0$ (i.e. the circuit is closed by using the switch B_2 in Fig. 160), the periodic motion will start as shown in Fig. 161. On the other hand if initially $v_0 = i_0 = 0$ (closing by the switch B_1) the performance is in accordance with

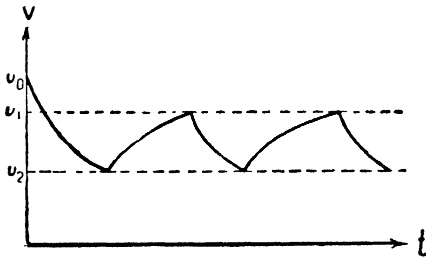


Fig. 161.

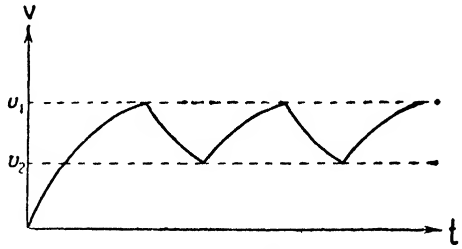


Fig. 162.

Fig. 162. After the system has reached a state compatible with the equation describing the system, further motion is uniquely determined by an equation of the first order together with the condition of jump.

We have found the qualitative features of the self-oscillations in the circuit containing a neon tube. In order to determine their quantitative characteristics we require the form of the nonlinear function $\phi(v)$. A very convenient approximation is to replace the graph of $\phi(v)$ by a broken line. This highly idealized characteristic still retains the fundamental traits of the real characteristic emphasized

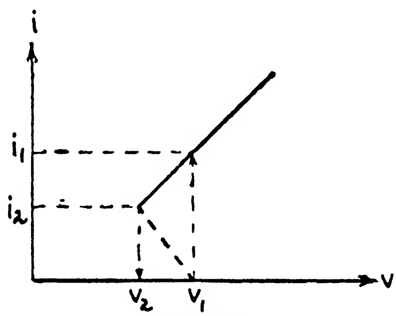


Fig. 163.

earlier, and consequently preserves the main properties of the neon tube. One analytic expression cannot cover the totality of this characteristic, and we have to proceed as we have done when examining discontinuous characteristics. Namely, we divide the whole region of possible motions into portions and for each of these portions we write an equation which is obtained for this portion. The investiga-

tion of the dropping portion "in front" of which passes the representative point (represented by a dotted line) can be omitted. The two other portions of the characteristic can be covered correspondingly by the equations:

1. $i = \phi(v) = 0$ when the tube is non-conducting,

2. $i = \phi(v) = \dfrac{v - v_0}{R_i}$ when the tube is conducting.

Let us start our investigation from the moment when the tube is non-conducting, i.e. when the first equation is valid, and when $v = v_2$. We obtain for the first portion

$$RC\dot{v} + v = E.$$

Since the initial condition is $v = v_2$ when $t = 0$, the solution is

$$v = E - (E - v_2)e^{\frac{-t}{RC}}. \tag{17}$$

When the voltage on the plates of the capacitor reaches the value v_1, the tube will fire and the current flowing through the tube will change abruptly. The current in the capacitance circuit will also change abruptly but in such a way that $R(I + i)$ and the voltage on the plates of the capacitor remain constant.

The initial value of v corresponding to the new equation which becomes valid after the tube has fired is the value v_1 for which the tube fires. After the firing the second equation of the characteristic becomes valid and the equation of motion is

$$RC\dot{v} + \frac{R}{R_i}(v - v_0) + v = E. \tag{18}$$

Setting

$$\rho = \frac{R_iR}{R + R_i},$$

the equation reduces to

$$C\dot{v} + \frac{v}{\rho} = \frac{E}{R} + \frac{v_0}{R_i}.$$

The solution of this equation for the initial conditions $v = v_1$ for $t = 0$ is

$$\frac{v}{\rho} = \frac{E}{R} + \frac{v_0}{R_i} + \left(\frac{v_1}{\rho} - \frac{v_0}{R_i} - \frac{E}{R}\right) \cdot e^{\frac{-t}{\rho C}}. \tag{19}$$

To calculate the period we may utilize the representations of $v(t)$ by (17) and (19) as follows. From (17) we find the time T_1 to charge the capacitor from v_2 to v_1 by substituting $v = v_1$ and taking logarithms. Thus

$$T_1 = RC \log \frac{E - v_2}{E - v_1}.$$

Similarly from (19), setting $v = v_2$, we find the time to discharge the capacitor from v_1 to v_2 as equal to

$$T_2 = \rho C \log \frac{(v_1 - v_0)R - (E - v_1)R_i}{(v_2 - v_0)R - (E - v_2)R_i}.$$

The period is then $T = T_1 + T_2$. Once we have T we may utilize the expressions (17) and (19), the first from 0 to T_1 and the second from T_1 to T, to expand $v(t)$ in Fourier series and thus have the spectral composition of the oscillations. As in all oscillations with a discontinuity of some sort the form is quite remote from the harmonic type, and this property is susceptible of wide applications in engineering.

§10. Relaxation Oscillations, Circuit Containing a Vacuum Tube

The so-called multivibrators provide another example of electrical self-oscillatory systems inducing relaxation oscillations. We have in

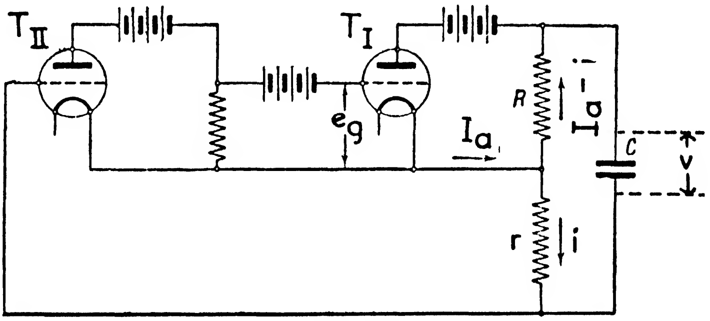


FIG. 164.

mind vacuum tube circuits without inductance or without capacitance. The simplest scheme of the first kind is represented in Fig. 164. We have examined similar schemes before. The second tube T_{II} in this scheme, as in other analogous schemes, serves only to provide the necessary sign for the voltage fed back by reversing the phase of the applied voltage. Therefore, as before, we shall assume that the characteristic of the tube T_{II} is linear and that the coefficient of amplifica-

tion given by this tube is a constant k_2. In practice this can be achieved by choosing an appropriate tube T_{II} whose saturation voltage is appreciably higher than the maximum voltage which can take place on the resistance r. We have then $e_g = k_2 r i$. Let the characteristic of the tube T_I be $I_a = \phi(e_g)$ where ϕ need not be specified for the present. Neglecting the reaction on the plate and the grid currents and transferring the origin of the coordinates to the point I_0 where I_0 is the constant component of the plate current of the tube T_I, we may describe the scheme under investigation by the following equations:

$$(r + R)i + v = R\phi(k_2 r i), \tag{20}$$

$$C\dot{v} = i. \tag{21}$$

Differentiating the first equation and eliminating $\dot{v}$, we obtain one differential equation of the first order:

$$[k_2 r R\phi'(k_2 r i) - (R + r)]\dot{i} = \frac{i}{C}. \tag{22}$$

To determine the periodic solutions one must make certain assumptions regarding the function $\phi(e_g)$. We assume that it has the ordinary

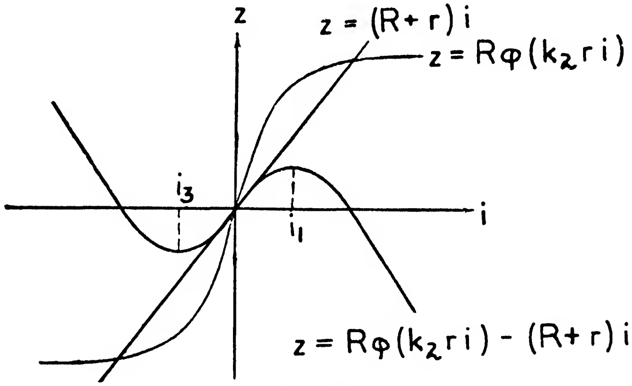


FIG. 165.

form and that the operating point ($i = 0$) corresponds to the maximum grid plate transconductance g_n, i.e. that $\phi'(0) = g_n$. In this case the equilibrium state $i = 0$ is unstable if

$$k_2 r R g_n > R + r, \tag{23}$$

for, according to (22), i and $\dot{i}$ have the same sign at the point $i = 0$. The function $v(i)$, defined by (20):

$$v(i) = R\phi(k_2 r i) - (R + r)i \tag{24}$$

can be represented by the graph of Fig. 165. This figure corresponds

to the case where condition (23) is fulfilled. It is easy to see that in this case i, i.e. $C\dot{v}$, is not a single-valued function of v. If $\dot{v}$ were single-valued, the possibility of periodic motions would be entirely excluded. For v is the voltage across C and consequently solutions

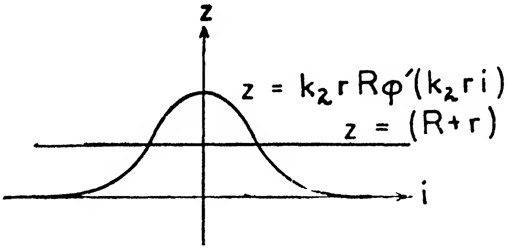


FIG. 166.

discontinuous in v are ruled out by our postulate. The function

$$z = k_2 r R \phi'(k_2 r i),$$

as well as the function $z = (R + r)$, is represented in Fig. 166, again for the case when the condition (23) is fulfilled. The graph shows that $P(i) = k_2 r R \phi'(k_2 r i) - (R + r)$ is a single-valued function of i, and therefore $\dot{i}$ is uniquely determined from

$$P(i)\dot{i} = \frac{i}{C}. \tag{25}$$

Thus, continuous periodic solutions are impossible for i, but i can change abruptly (since the circuit has no inductance) and one may have

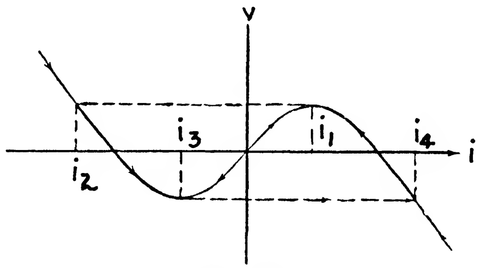


FIG. 167.

discontinuous periodic solutions. To find them we shall first examine the graph in the (v,i) plane. We can consider the curve defined by (24) in Fig. 165 as the path of our system. The representative point moves along this curve with finite velocity. The direction of its motion along the curve is determined by the sign of $\dot{i}$, i.e. by the sign of $P(i)$ in (25), and is shown by arrows in Fig. 167. At the points

where $P(i) = 0$, i.e. at the extreme points of the curve, $\dot{i} = \infty$ and consequently at these points a jump is possible. As in the previous case, these points do not correspond to the equilibrium state, although the representative point moves toward them on both sides. During the jump, v must remain constant; and the position to which the point jumps is uniquely determined by the condition of jump, since it asserts that the jump must be parallel to the i-axis (dotted arrows in Fig. 167). After the jump, the representative point moves again along the curve in the direction represented by the arrows. It is easy to see that the representative point will return to its initial position and that consequently the system experiences a periodic process.

The oscillations which are established in the system consist of two continuous motions, from i_2 to i_3 and from i_4 to i_1, and of two jumps

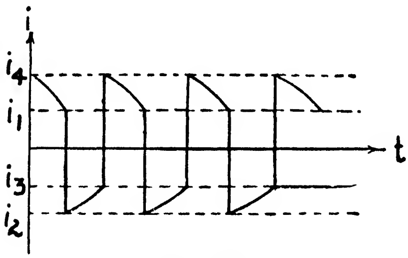


FIG. 168.

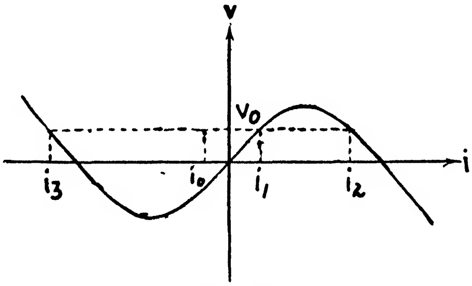


FIG. 169.

from i_1 to i_2 and from i_3 to i_4. To the periodic motion there corresponds a "closed" path consisting of two arcs of the v,i curve, whose extremities are "connected" by the jumps. The form of the oscillations, i.e. the form of the function $i = \phi(t)$ is shown in Fig. 168.

Thus, the periodic process in the system is uniquely defined. However, here, as in the two preceding examples, we are not in a position to tell how the oscillations are started. For example if the initial position is (i_0,v_0) in Fig. 169, then the representative point is equally justified in jumping to (i_1,v_0), (i_2,v_0) or (i_3,v_0). The question can be settled either by a certain additional assumption or by taking into account one of the parameters that we have neglected, for example inductance, and investigating the appropriate system of two equations of the first order. In general, the presence of inductance causes the current to change gradually and not abruptly (although with high velocity), the appearance of the e.m.f. of self-induction causes the voltage on the plates of the capacitor to change and therefore during the "jump" the representative point will move along certain curves and not along straight lines parallel to the i-axis. There results a

path such as in Fig. 170. Its form and the way in which the representative point reaches it, if it has not started on it, cannot be rigorously determined on the basis of our idealization. In the next chapter

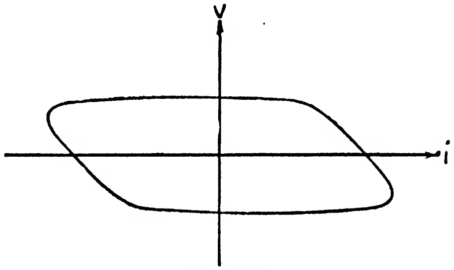


Fig. 170.

we shall give a rigorous, although only qualitative, answer to this question.

It is easy to see that the position to which the representative point jumps at the initial moment affects the character of the motion only at the very beginning, and in fact usually it is unimportant. All the

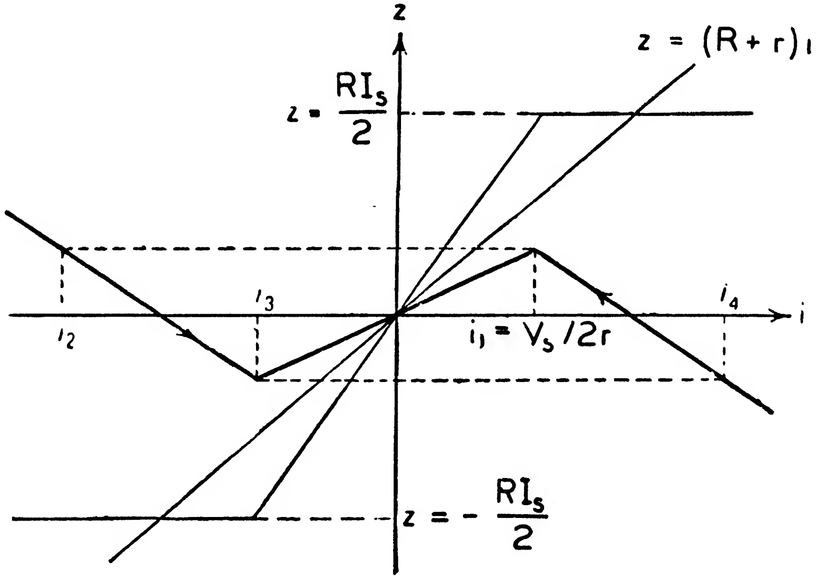


Fig. 171.

other questions arising from the study of stationary motions can be dealt with, and we can calculate period and amplitude if we have an explicit expression for $\phi(e_g)$. In fact, from (25) we find

$$dt = C\frac{P(i)}{i} \cdot di$$

and hence for the period

$$T = C\int_{i_2}^{i_3} \frac{P(i)\,di}{i} + C\int_{i_4}^{i_1} \frac{P(i)\,di}{i}.$$

This time again a broken-line approximation as in Fig. 171 yields a very satisfactory expression:

$$T = 2C(R + r)\log\left(\frac{2I_sRr}{V_s(R + r)} - 1\right),$$

where I_s and V_s are the current and voltage at saturation. This expression shows that the frequency of oscillations increases rapidly when the excitation limit is approached. The relaxation oscillations of the present scheme are characterized by the fact that they always go much

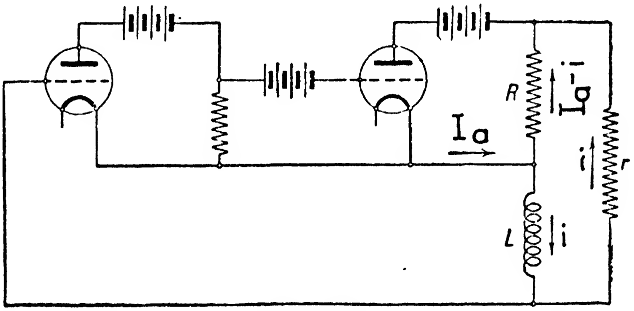


FIG. 172.

beyond the saturation voltage and so the form of the "exterior" portions of the characteristic substantially affects the process. Therefore it is not convenient to represent approximately the characteristic by a polynomial for which saturation is always followed by a dropping portion of the curve, whereas with relaxation oscillations the characteristics after saturation and before the appearance of the initial current are represented by a horizontal line.

A self-oscillatory system without capacitance is represented in Fig. 172. Under the same conditions as previously, we obtain this time the following equation of the first order (i cannot jump):

(26) $$L\dot{i} + (r + R)i - R\phi(k_2L\dot{i}) = 0.$$

Setting $\dot{i} = y$ and differentiating this equation we obtain again an equation of the first order (here y can have jumps):

(27) $$L(1 - k_2R\phi'(k_2Ly))\dot{y} + (r + R)y = 0.$$

These equations are analogous to equations (22) and (24) examined

before. One of these equations (26) defines $\dot{i}$ as a non-single-valued function of i and admits continuous solutions for i. Equation (27) defines $\dot{y}$ in single-valued manner, but the system admits discontinuous solutions for y. As in the previous case, we can find periodic solutions continuous in i and discontinuous in $\dot{i}$.

Finally, the scheme of Fig. 173 represents a case where periodic motion cannot take place in a system described by non-linear equations of the first order. Making the same assumption as before with regard

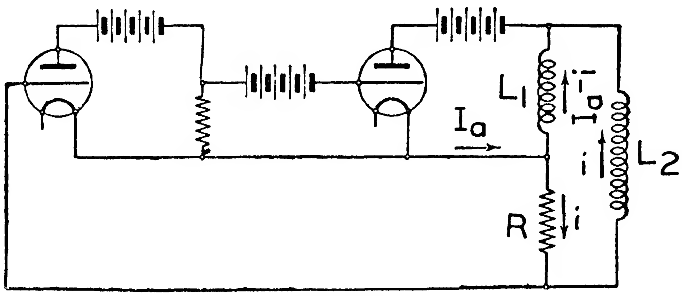


FIG. 173.

to the characteristic and neglecting the current in the circuit and the plate reaction, we find

$$L_1\phi(k_2Ri) = (L_1 + L_2)\dot{i} + Ri \tag{28}$$

or

$$\left(Rk_2\phi'(k_2Ri) - \frac{L_1 + L_2}{L_1}\right)\dot{i} = \frac{R}{L_1}\,i. \tag{29}$$

If $Rk_2\phi'(0) - [(L_1 + L_2)/L_1] > 0$, the unique equilibrium state is unstable and the system leaves it. Since (29) determines $\dot{i}$ as a single-valued function of i, continuous solutions are impossible for i. On the other hand, the presence of inductance rules out discontinuous solutions for i. Thus, if the equilibrium state is unstable, the system leaves it with an aperiodic motion and passes into regions where it cannot be described by (28), for example into the region where, due to the e.m.f. of self-induction, the resulting plate voltage is distinctly different from the static voltage and therefore the effect of this voltage on the intensity of the plate current has to be taken into account. In fact, if we take into consideration the reaction on the plate, we arrive at an equation of the second order which admits continuous periodic solutions.

§11. VOLTAIC ARC IN PARALLEL WITH CAPACITANCE

We will conclude with an electrical system with several states of equilibrium, in which periodic oscillations cannot take place. This example will illustrate even more clearly the assumptions regarding

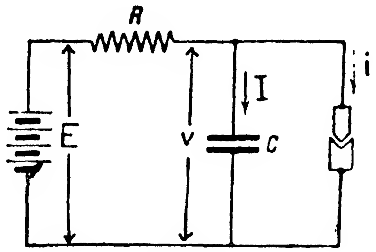


FIG. 174.

the possibility and the necessity of jumps, stated in the preceding chapters. The system consists of a voltaic arc connected in parallel with a capacitance C and a resistance R. Let $i = \phi(v)$ be the non-linear relation between the voltage v applied to the arc and the current i in the arc. This "characteristic of the arc" is shown in Fig. 175.

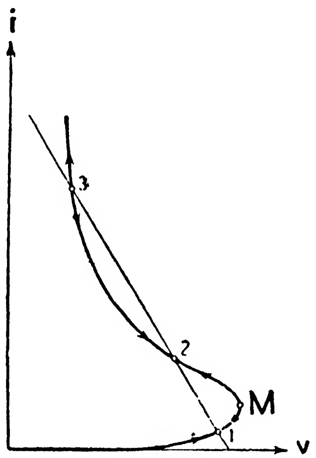


FIG. 175.

Designating the current in the circuit containing capacitance by I, we obtain by Kirchhoff's law:

$$R[\phi(v) + I] + v = E, \qquad I = C\dot{v}.$$

Eliminating I, we have one non-linear differential equation of the first order:

$$\dot{v} = f(v) = \frac{E - v - R\phi(v)}{RC},$$

where the function $f(v)$ is not single-valued. The equilibrium state is defined by the condition $f(v) = 0$ or by $E - v - R\phi(v) = 0$. In order to find the roots of this equation we draw the curve $Z = \phi(v)$ and the straight line $Z = (E - v)/R$ and find their intersection points (Fig. 175). We shall first investigate the case where the R is such that there are three equilibrium states (three intersection points of the curve and the line). By subtracting the ordinates of the curve from those of the line we obtain the graph of $f(v)$ (Fig. 176). We can take the curve $\phi(v)$ or, better, the curve $f(v)$ as the path. The direction of motion of the representative point along the curve $f(v)$ is shown

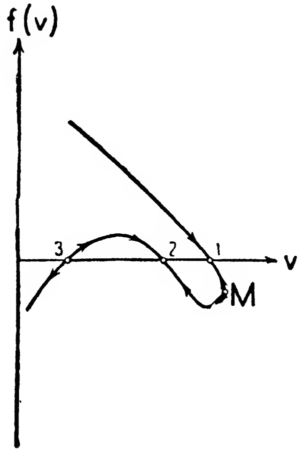


FIG. 176.

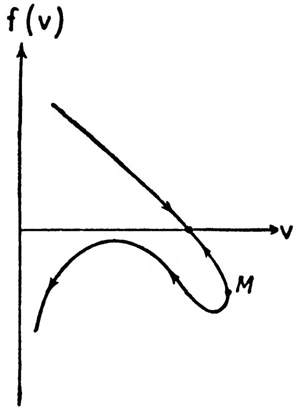


FIG. 177.

by the arrows in Fig. 176. It is governed by the sign of $\dot{v} = f(v)$. That is: when $f(v) > 0$ the point moves towards large v, and when $f(v) < 0$ the point moves toward small v. We can see immediately that the points 1 and 2 are stable and the point 3 unstable. Since $\dot{i} = \dot{\phi} = \phi'(v)\dot{v} = \phi'(v)f(v)$, at the point where $\phi'(v) = \pm\infty$ (at the same point $f'(v) = \mp\infty$, since $-1 - R\phi'(v) = RCf'(v)$) the current changes, or, what is the same, the vertical component of the velocity of the representative point on the path also goes to infinity. Consequently the point M on Figs. 175 and 176 is the point of unlimited acceleration. However, a jump out of the point of unlimited acceleration M is altogether impossible because to one value of v there corresponds only one value of $\dot{v}$; consequently the condition of jump cannot be satisfied. Although $f(v)$ is a non-single-valued function of v, the function is such that periodic motions are impossible. This is quite natural since the non-single-valuedness of the function $f(v)$ is only

necessary but not sufficient for the existence of continuous periodic solutions for the equation $\dot{v} = f(v)$.

When R increases, the straight line $Z = (E - v)/R$ turns counterclockwise around the intersection point with the horizontal axis, the two neighboring equilibrium states (the stable and the unstable) approach each other and finally coincide. The direction of motion of the representative point is then shown in Fig. 177.

It is necessary to keep in mind that there is a limit to the magnitude of the current through the arc. When the current increases greatly,

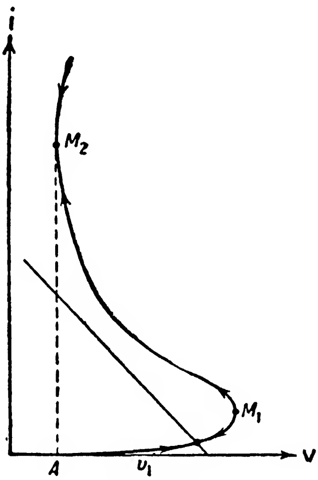


FIG. 178.

the characteristic of the arc adopted by us becomes incorrect in the sense that it does not reflect the properties of the arc for a very high current. Let us assume that the resistance of the arc again becomes positive when the current i is very high, i.e. $\dot{v} < 0$. From the point M_2 a jump is not only possible but inevitable; the representative point jumps to the point A on another branch and the arc is extinguished. After that the capacitor is being charged again until the voltage on its plates reaches the value v_1 for which the arc fires. Then the representative point moves toward the unique stable equilibrium state and will stop there. As in the case of a mechanical system examined before, the system experiences a jump which takes it to a stable equilibrium state instead of generating a periodic motion.

Regarding the location of equilibrium states and that of the points of unlimited acceleration on the path, we may say the following: the direction of motion of the representative point changes only either at equilibrium states (where $\dot{v}$ and $\ddot{v}$ change their sign) or at points of

unlimited acceleration (where only $\dot{v}$ changes its sign). Consequently the equilibrium states and the points of unlimited acceleration are always so situated that the stable points alternate with the unstable. The "stable" and "unstable" points can be either equilibrium states or points of unlimited acceleration. Thus, in the example just examined, between two stable equilibrium states there is one "unstable" point of unlimited acceleration. We must remember that the points of unlimited acceleration were obtained as a result of idealization—neglecting the mass or the inductance. If we take into account the mass or inductance, these points of unlimited acceleration disappear while the equilibrium states remain. This does not cause any difficulty since we have then an equation of the second order and the representative point can then move over the entire phase plane. The investigation of systems described by equations of the second order enables us to show more precisely that, when one of the oscillating parameters tends to zero, the motion over the phase plane goes into a more special motion along the only remaining path. Let us note that when degeneracy takes place, the equilibrium states may remain but their character may change. For example, unstable equilibrium states can become stable. All the questions relative to the passage from a more general case to a particular degenerate one will be examined in Chap. VII.

CHAPTER V

Dynamical Systems Described by Two Differential Equations of the First Order

§1. Introduction

In this chapter we turn our attention to autonomous dynamical systems represented by two equations of the first order

$$(1) \qquad \dot{x} = P(x,y), \qquad \dot{y} = Q(x,y).$$

We have seen repeatedly how such systems arise and we shall now deal with their general theory. The term "autonomous" refers to the fact that the right hand sides do not contain the time. In many cases the x,y plane is a phase plane and so we shall continue to call it that, and refer to the point $M(x,y)$ as the representative point of the system. The vector (P,Q) will be referred to as the velocity of M.

Let us suppose that $x = f(t)$, $y = g(t)$ is a solution or "motion." Then whatever fixed τ we have

$$\frac{df(t+\tau)}{d(t+\tau)} = \frac{df(t+\tau)}{dt} = P(f(t+\tau),g(t+\tau))$$
$$\frac{dg(t+\tau)}{d(t+\tau)} = \frac{dg(t+\tau)}{dt} = Q(f(t+\tau),g(t+\tau)).$$

Hence $x = f(t+\tau)$, $y = g(t+\tau)$ is likewise a motion. It is evident however that the two motions are along the same curve γ in the x,y plane, each describing it merely at different times. We refer to the curve γ as a *path*. Our problem may be described as the investigation of phase portraits, i.e. of families of paths of systems (1).

A simple example will clarify the distinction between motions and paths. Consider a railroad with a given line γ between two cities. Let the railroad have a fixed daily schedule such that the same train, assimilated to a material point, describes the line γ once every day. The motions are the trips made by the train and the line γ is a path.

Returning to our main question, the most salient features of the system (1) are: (a) the *singular* points, also called *equilibrium points* or *states* and they are the intersections of the curves $P = 0$, $Q = 0$ (points of velocity zero), and (b) the isolated *closed* paths or *limit-*

cycles in Poincaré's terminology which correspond to isolated periodic motions.

There is manifestly no reason to expect systems (1) of arbitrary generality in the physical applications. Owing to this we shall only consider systems (1) in which P and Q are real polynomials without common factor. This means in particular that the curves $P = 0$, $Q = 0$ have only isolated intersections. Cases may well arise in practice where this does not hold. Consider for instance a system

$$(1a) \qquad \begin{cases} \dot{x} = P(x,y)R(x,y) \\ \dot{y} = Q(x,y)R(x,y) \end{cases}$$

where P,Q are polynomials without common factor, and R a polynomial or quotient of two polynomials. Introduce a new time variable τ by $d\tau = R\,dt$, and consider the system

$$(1b) \qquad \frac{dx}{d\tau} = P(x,y), \qquad \frac{dy}{d\tau} = Q(x,y).$$

Outside of the curve $R = 0$, the two systems (1a) and (1b) have the same paths, but the corresponding motions are not the same. One would replace (1a) by (1b) and from the paths of (1b) deduce those of (1a). The sense of description of the paths in (1a) would have to be obtained directly from (1a) itself but this would cause no difficulty. We shall therefore confine the theory to the case where the polynomials P and Q have no common factor.

In practice all systems (1) are approximated and hence replaced by the special systems in which P and Q are polynomials. Thus as a general rule the polynomials P,Q are only approximations to certain empirical expressions and their coefficients are only known with a certain approximation. Whatever is significant must then remain unchanged when the coefficients vary slightly. We make therefore the very natural assumption that the qualitative nature of the phase portrait is unchanged under a small variation of the coefficients of the polynomials P,Q: the variation of each coefficient does not exceed in absolute value a certain positive ϵ. We shall refer to this property as *structural stability.* As we shall see it has certain very definite simplifying consequences regarding singular points, limit-cycles, and the limits of paths, which we shall take up as we come to them.

Historically, the treatment of the behavior of the paths about the singular points is due to Poincaré and Liapounoff, the more complete treatment being Liapounoff's. On the other hand Poincaré was the

first to investigate the phase portrait, and most results and methods go back to him, although important complements are due to Bendixson. The notion of *structural stability* (under another name) was introduced and thoroughly discussed, under much broader assumptions than we have made here, in a paper by Andronow and Pontrjagin.[1]

We recall *Cauchy's existence theorem:* together with its complements for analytic functions it asserts here that there exists a unique solution

$$x = f(x_0,y_0,t), \qquad y = g(x_0,y_0,t)$$

such that for $t = 0$ we have $x = x_0$, $y = y_0$, and that f,g are analytic in all three arguments about $x_0,y_0,0$. Then

$$x = f(x_0,y_0,t - t_0), \qquad y = g(x_0,y_0,t - t_0)$$

is the (unique) solution such that $x = x_0$, $y = y_0$ for $t = t_0$. The two solutions just considered correspond to the same path γ through the point $M(x_0,y_0)$.

Since P and Q are polynomials, the solutions in question may be extended indefinitely for both $t \to +\infty$ and $t \to -\infty$. We have thus two halves of paths separated by M, the one resulting from the extension for increasing t and the other from the extension for decreasing t and they are referred to as *positive half-path* and *negative half-path* determined by M, written γ_M^+, γ_M^-. The two together, with M included, form the complete path γ.

As a consequence of the existence theorem we may state the following properties:

I. Through every point of the plane there passes one and only one path. That is to say, paths cannot cross one another.

II. A singular point $A(a,b)$ is evidently a path. We may refer to it as a point-path. By **I** no other path than A itself may pass through A. Hence if an ordinary path γ tends to A, through increasing time, it cannot reach it in a finite time. Similarly if γ tends to A, through decreasing time, upon being described backwards it will not reach A in a finite time. This may be described more loosely as: a path may tend to a singular point A but never reach it.

§2. Linear Systems

We shall first discuss the simplest dynamical systems of type (1), namely those which, with an adequate idealization are represented by two linear equations

$$\dot{x} = ax + by, \qquad \dot{y} = cx + dy, \tag{2}$$

where a,b,c,d; are real constants.

[1] *Comptes Rendus*, Acad. Sc. U.S.S.R., Vol. 14, 1937, pp. 247–250.

As we shall see the nature of the phase portrait is wholly governed by the nature of the roots S_1,S_2 of the *characteristic equation*

$$\begin{vmatrix} a - S, & b \\ c, & d - S \end{vmatrix} = 0$$

or in expanded form

$$S^2 - (a + d)S + ad - bc = 0. \tag{3}$$

We refer to S_1,S_2 as the *characteristic roots.*

Without giving any details we may say here and now that if one of the following circumstances arises: one of the roots is zero, both are pure imaginary, or the two roots are equal, the paths behave in a qualitative manner which is distinct from what happens when these accidents do not occur, since each imposes a special relation upon the coefficients a,b,c,d. To take one of the cases: if the roots are imaginary their common real part is $\frac{1}{2}(a + d)$. Hence they will be pure imaginary only if $a + d = 0$. Similarly for the other two special circumstances. Now if we change ever so slightly the numbers a,b,c,d these special relations will cease to hold, and the portrait of the family of paths will have been changed. Since only systems with stable structure are to be admitted, we may rule out at the outset the three special cases. That is to say we may assume that S_1,S_2 are neither *zero nor pure imaginary and that they are distinct.*

Since S_1,S_2 are distinct the general solution of the system may be obtained in the following way:

I. *Suppose* $b = c = 0$. Then $S_1 = a$, $S_2 = d$ and the system is in the so-called *canonical form*

$$\dot{x} = S_1x, \qquad \dot{y} = S_2y \tag{4}$$

whose general solution is

$$x = C_1e^{S_1t}, \qquad y = C_2e^{S_2t}. \tag{5}$$

II. *Suppose b and c not both zero.* Let say $b \neq 0$. Then we can solve for x in the form

$$x = C_1e^{S_1t} + C_2e^{S_2t}, \tag{6}$$

and obtain y by substitution in the first equation (2) as

$$y = C_1k_1e^{S_1t} + C_2k_2e^{S_2t}, \tag{7}$$

$$k_1 = \frac{S_1 - a}{b}, \qquad k_2 = \frac{S_2 - a}{b}. \tag{8}$$

It may be observed that k_1,k_2 are the roots of the quadratic equation

$$bk^2 + (a - d)k - c = 0. \tag{9}$$

It is obtained by noticing that if S is any characteristic root then the corresponding number k is such that $S = bk + a$. By substituting this value for S in the characteristic equation (3) we obtain (9).

We are particularly interested in the nature of the paths. It turns out that to investigate them adequately a very convenient process consists in changing coordinates. It will be recalled that a linear transformation of coordinates

$$\xi = \alpha x + \beta y, \qquad \eta = \gamma x + \delta y \tag{10}$$

has merely the effect of distorting figures, replacing for instance a family of concentric circles by a family of similar concentric ellipses (with the same axes of symmetry). Thus the general aspect of the characteristics may equally well be obtained by means of the ξ,η coordinates, i.e. relatively to the ξ,η plane. Let us endeavor in particular to find a transformation (10) such that in the new coordinates the system will be in the canonical form (4), i.e. that

$$\dot{\xi} = S_1\xi, \qquad \dot{\eta} = S_2\eta, \tag{11}$$

where for the present S_1 and S_2 are merely some constants. Differentiating the transformation formulae (10), we find

$$\dot{\xi} = \alpha\dot{x} + \beta\dot{y}, \qquad \dot{\eta} = \gamma\dot{x} + \delta\dot{y}.$$

Hence if (11) is to hold we must have identically

$$\begin{cases} S_1(\alpha x + \beta y) = \alpha(ax + by) + \beta(cx + dy), \\ S_2(\gamma x + \delta y) = \gamma(ax + by) + \delta(cx + dy). \end{cases} \tag{12}$$

By comparing the coefficients of x and y on both sides we obtain two pairs of equations linear and homogeneous in α,β in the first pair and in γ,δ in the second:

$$\begin{cases} \alpha(a - S_1) + \beta c = 0 \\ \alpha b + \beta(d - S_1) = 0 \end{cases} \tag{13}$$

$$\begin{cases} \gamma(a - S_2) + \delta c = 0 \\ \gamma b + \delta(d - S_2) = 0. \end{cases} \tag{13a}$$

These equations will yield for (α,β) and (γ,δ) solutions which are not both zero when and only when the determinants of the coefficients are zero, i.e. when and only when S_1 and S_2 are the roots of the equation:

$$S^2 - (a + d)S + (ad - bc) = 0,$$

i.e. roots of the characteristic equation. Each pair of equations (13) determines only the ratio of the unknowns. The first pair determines α/β and the second γ/δ. Since, according to our assumptions $S_1 \neq S_2$, these ratios can be chosen to be different, and consequently, the determinant

$$\begin{vmatrix} \alpha & \beta \\ \gamma & \delta \end{vmatrix}$$

will not be zero. Hence one may solve equations (10) for x and y and consequently we have a one-to-one transformation. Thus we can see that in the non-degenerate cases it is always possible to transform our system into canonical form. Let us examine the different possible cases.

1. The roots S_1 and S_2 are real and of the same sign. Then the transformation coefficients are real and we have a mapping from a real plane (x,y) to a real plane (ξ,η). In the transformed phase plane (11) holds so that it is easily dealt with. We will then have to interpret the results obtained for the initial (x,y)-plane. Dividing one canonical equation by the other we have:

$$\frac{d\eta}{d\xi} = \frac{S_2}{S_1}\frac{\eta}{\xi},$$

which by integration yields

$$\eta = C\xi^m, \qquad m = \frac{S_2}{S_1}.$$

Since S_1 and S_2 have the same sign, $m > 0$, and the curves are of parabolic type. If $m > 1$, the paths are all (except the curve $C = \infty$ corresponding to the η axis) tangent to the ξ axis (Fig. 179). If $m < 1$, the paths are all (except the curve $C = 0$, corresponding to the ξ axis) tangent to the η axis (Fig. 180). The origin is the singular point, and it is a node.

It is easy to determine the direction of motions in the phase plane. If S_1 and S_2 are negative, then by (11), ξ and η decrease with time. As t increases the representative point approaches the origin without ever reaching it. We have, therefore, a *stable node*. If S_1 and S_2 are positive, ξ and η will increase with time. The representative point departs from the origin and we have an *unstable node*. Let us now return to the (x,y)-plane. As we know, the phase portrait does not change, but the tangents to the parabolic curves at the singular

point will no longer coincide with the coordinate axes. It becomes then of interest to determine their direction. In the (ξ,η)-plane these tangents were represented by the axes and therefore, we must determine the straight lines in the (x,y)-plane corresponding to the straight

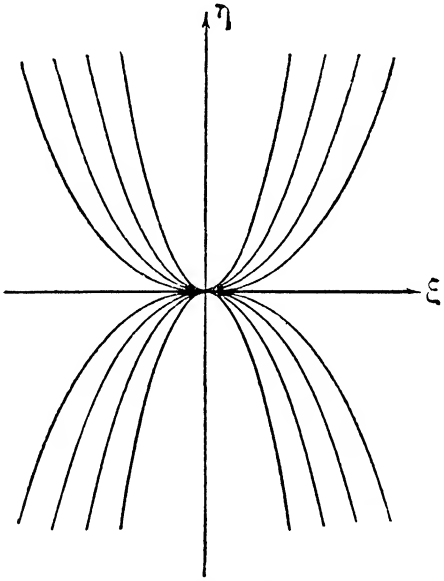


Fig. 179.

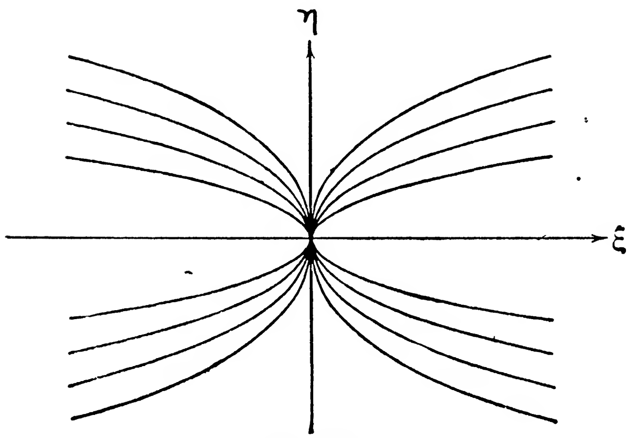


Fig. 180.

lines $\xi = 0$ and $\eta = 0$. Equation (10) shows that to the line $\xi = 0$ corresponds the line:

$$\alpha x + \beta y = 0 \qquad \text{or} \qquad y = \frac{-\alpha x}{\beta}$$

i.e. the line through the origin with slope

$$k_1 = \frac{-\alpha}{\beta} = \frac{c}{a - S_1} = \frac{d - S_1}{b},$$

while the line $\eta = 0$ corresponds in the (x,y)-plane to the line

$$\gamma x + \delta y = 0 \qquad \text{or} \qquad y = -\frac{\gamma}{\delta}x,$$

i.e. to a line through the origin with slope

$$k_2 = -\frac{\gamma}{\delta} = \frac{c}{a - S_2} = \frac{d - S_2}{b}.$$

From the relations $k = (d - S)/b$ or $S = d - bk$, we find by substituting in (3) that k_1, k_2 are again the roots of (9) and so they are the same k_1, k_2 as before.

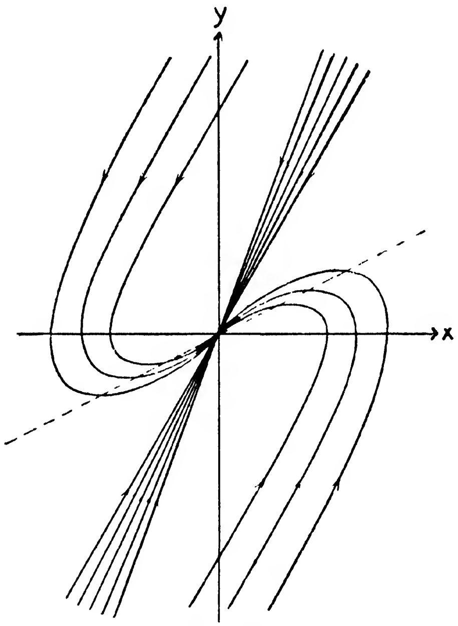


FIG. 181.

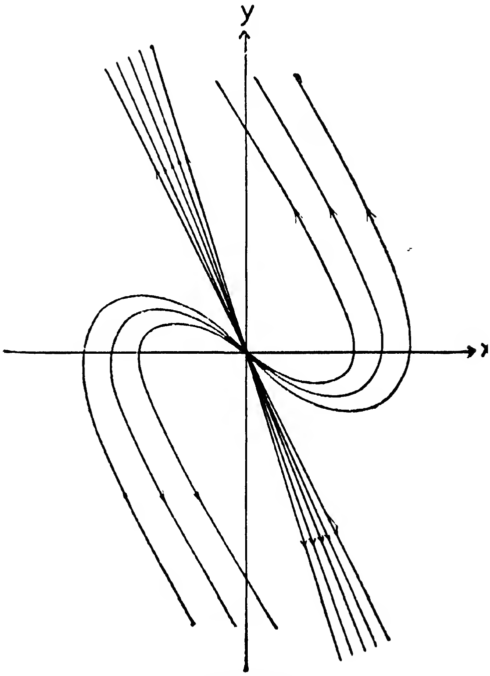


FIG. 182.

The lines $y = k_1x$ and $y = k_2x$ are on the one hand solutions of the equation $dy/dx = (cx + dy)/(ax + by)$ (in the same way as $\xi = 0$ and $\eta = 0$ represent solutions of $d\eta/d\xi = a\eta/\xi$), and on the other hand they are the tangents at the origin of the coordinates to all the solutions.

It is now easy to represent directly the paths in the case of the stable node (Fig. 181) or the unstable node (Fig. 182).

2. The roots S_1 and S_2 are real but of opposite signs. This time setting $m = -S_2/S_1 > 0$, we find as before

$$\eta = C\xi^{-m}.$$

This equation determines a family of curves of hyperbolic type having the axes for asymptotes. When $m = 1$ we have a family of ordinary hyperbolas with symmetric branches. Here the halves of the axes of coordinates are paths. The origin is of course the singular point. Singular points of this type are called "saddle points." The same considerations as before enable us to determine the character of the motion of the representative point. Let us assume for example that

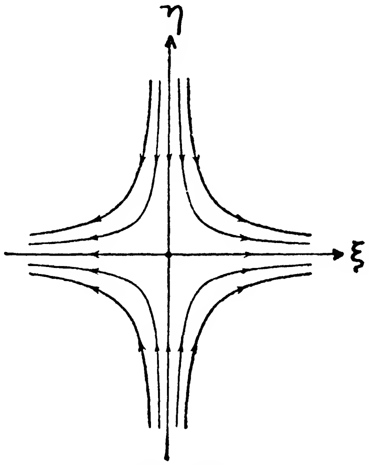


FIG. 183.

$S_1 > 0$ and $S_2 < 0$. Then the representative point on the ξ-axis will depart from the origin of the coordinates, while on the η-axis it will approach the origin without ever reaching it. The direction of the motions along the other paths can be easily found by continuity (Fig. 183). As we know, a saddle point is unstable. If we return now to the coordinates x,y we obtain the same qualitative picture for the paths. As in the preceding case, the slopes of the straight lines passing through the singular point are the roots of (9).

3. S_1 and S_2 are complex conjugates. It is easy to see that for x and y real, ξ and η can be taken complex conjugates. If, however, we introduce an additional transformation, one may also reduce this to a real linear homogeneous transformation. Let us assume:

$$(14) \qquad \begin{cases} S_1 = a_1 + ib_1, & \xi = u + iv, \\ S_2 = a_1 - ib_1, & \eta = u - iv, \end{cases}$$

where a_1, b_1, u, v are real. It is possible to show that the transformation from x,y to u,v is, according to our assumptions, real, linear, homogeneous, with a non-zero determinant. On the basis of equations (14) we have:

$$\text{(15)} \qquad \begin{cases} \dot{u} + i\dot{v} = (a_1 + ib_1)(u + iv), \\ \dot{u} - i\dot{v} = (a_1 - ib_1)(u - iv), \end{cases}$$

and consequently

$$\text{(16)} \qquad \dot{u} = a_1 u - b_1 v, \qquad \dot{v} = a_1 v + b_1 u.$$

In order to investigate this system, let us first examine the paths in

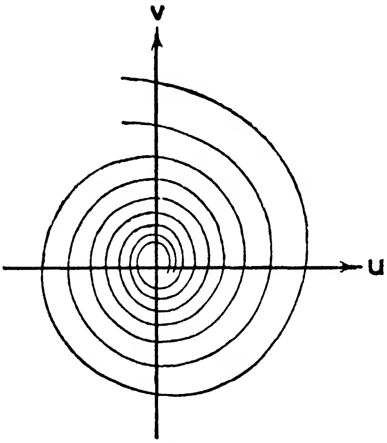


Fig. 184.

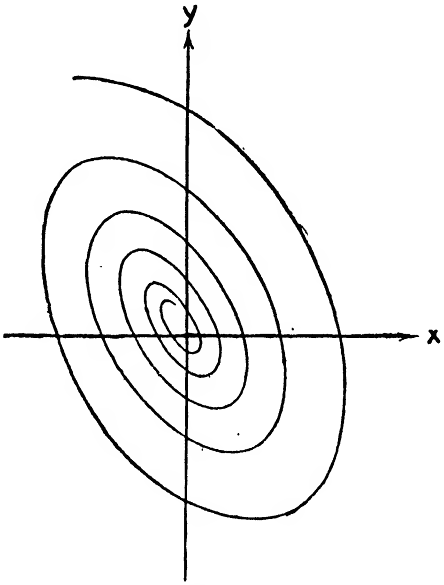


Fig. 185.

the (u,v)-plane. Their differential equation is

$$\frac{dv}{du} = \frac{a_1 v + b_1 u}{a_1 u - b_1 v}.$$

It is much easier to investigate by passing to polar coordinates. Setting $u = r \cos \phi$, $v = r \sin \phi$, we find in fact

$$\frac{dr}{d\phi} = \frac{a_1}{b_1} r,$$

and therefore $r = ce^{\frac{a_1}{b_1}\phi}$. Hence in the (u,v)-plane we have a family of logarithmic spirals asymptotic to the origin. The origin is thus a focus. Let us determine the character of motion of the representative

point along the curves. Multiplying the first of the equations (16) by u and the second by v we obtain

$$\tfrac{1}{2}\dot{\rho} = a_1\rho, \qquad \rho = r^2 = u^2 + v^2.$$

Thus when $a_1 < 0$ (a_1 = the real part of S_1 and S_2) the representative point approaches the origin continuously without ever reaching it. Therefore when $a_1 < 0$, we have a *stable focus*. If $a_1 > 0$ the representative point departs from the origin and we have an *unstable focus*. When we pass from the (u,v)-plane to the initial (x,y)-plane the spirals remain spirals but they become deformed.

To sum up, we have the following five possibilities:

1. Stable node (S_1 and S_2 are real and negative).
2. Unstable node (S_1 and S_2 are real and positive).
3. Saddle point (S_1 and S_2 are real and have different signs).
4. Stable focus (S_1 and S_2 are complex and their real part is negative).
5. Unstable focus (S_1 and S_2 are complex and their real part is positive).

Certain properties of the roots are recalled for further reference. Set

$$p = -(a + d), \qquad q = ad - bc \tag{17}$$

so that the characteristic equation becomes

$$S^2 + pS + q = 0. \tag{18}$$

As is well known

$$S^2 + pS + q \equiv (S - S_1)(S - S_2).$$

Expanding the right-hand side and identifying the coefficients of the powers of S on both sides we obtain the classical relations for the sum and product of the roots

$$S_1 + S_2 = -p, \qquad S_1S_2 = q, \tag{19}$$

which we will find highly convenient. Side by side with them we recall that the roots are imaginary when and only when

$$\delta = p^2 - 4q < 0,$$

their values being then

$$S_1 = \frac{-p + i\sqrt{-\delta}}{2}, \qquad S_2 = \frac{-p - i\sqrt{-\delta}}{2}.$$

We note the following properties:

I. $-p =$ twice the common real part of the roots.

II. If the roots are complex their product is of the form $(\alpha + i\beta) \cdot (\alpha - i\beta) = \alpha^2 + \beta^2 > 0$. Hence $q > 0$.

III. If $q < 0$ the roots are real and of opposite sign and conversely. Hence $q < 0$ is a n.a.s.c. for a saddle point.

IV. The relations $p > 0$, $q > 0$ are:

necessary for a stable focus (complex roots with negative real part);

necessary for a stable node (real negative roots);

n.a.s.c. for stability: there will be a stable node if $\delta > 0$, a stable focus if $\delta < 0$.

A noteworthy diagram may also be drawn to represent the different possibilities by taking p,q as rectangular coordinates. The first

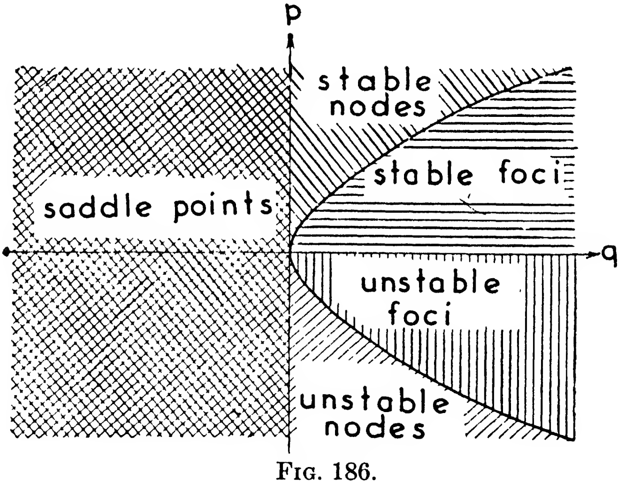


FIG. 186.

quadrant corresponds to stability. The regions of nodes and foci are separated by the parabola $p^2 = 4q$. The saddle points correspond to the region to the left of the vertical axis.

§3. An Example: The Universal Circuit

The circuit of Fig. 187 (investigated in this connection by Chaikin) when suitably idealized and, in particular, "linearized" may serve to illustrate the considerations developed for general linear systems. We shall assume that the characteristics of both tubes are linear. This assumption, as already observed, has meaning only within a narrow range of variation of the voltage on the grids of the tubes, and therefore

the linearization prevents us from investigating the behavior of the system throughout the whole range of the variables. Besides "linearization" we shall apply another idealization in that we shall neglect the grid current. Then using Kirchhoff's law, we can describe the behavior of the system by the following two equations:

$$r_2 i_2 = -\int \frac{i_2}{C_2}\,dt + \int \frac{i_1 - i_2}{C_1}\,dt,$$

$$r_1 i_1 - R(I_a - i_1) = -\int \frac{i_1 - i_2}{C_1}\,dt.$$

Setting $I_a = gk(r_1 i_1 + r_2 i_2)$ where g is the grid plate transconductance characteristic of the tube I, and k is the amplification coefficient of

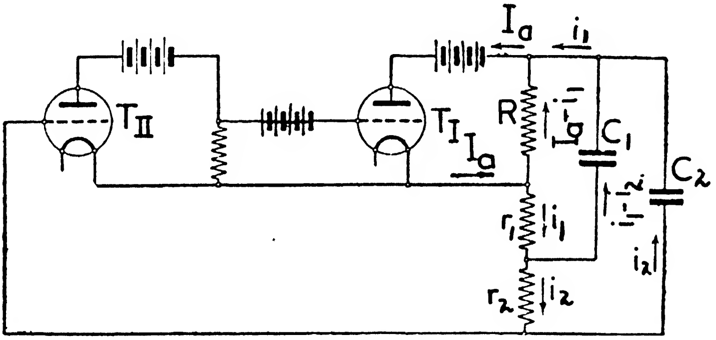


FIG. 187.

the stage with the tube II, we obtain two differential equations of the first order:

$$\dot{i}_1 = \frac{\dfrac{-1}{C_1}(1 - Rgk)i_1 + \left(\dfrac{1}{C_1}(1 - Rgk) - \dfrac{1}{C_2}Rgk\right)i_2}{R + r_1(1 - Rgk)}$$

$$\dot{i}_2 = \frac{\dfrac{1}{C_1}i_1 - \left(\dfrac{1}{C_1} + \dfrac{1}{C_2}\right)i_2}{r_2}$$

or

$$\dot{i}_1 = -\frac{\alpha}{C_1(R + \alpha r_1)}\,i_1 + \left(\frac{\alpha}{C_1(R + \alpha r_1)} - \frac{1 - \alpha}{C_2(R + \alpha r_1)}\right)i_2$$

$$\dot{i}_2 = \frac{1}{C_1 r_2}\,i_1 - \frac{1}{r_2}\left(\frac{1}{C_1} + \frac{1}{C_2}\right)i_2$$

where $\alpha = 1 - Rgk$ is either negative or positive and $< +1$. For the

sake of simplicity let us assume that $C_1 = C_2 = C$. Then the system becomes

$$(20)\qquad \begin{cases} \dot{i}_1 = \dfrac{1}{C(R + \alpha r_1)}(-\alpha i_1 + (2\alpha - 1)i_2) \\ \dot{i}_2 = \dfrac{1}{Cr_2}(i_1 - 2i_2). \end{cases}$$

The characteristic equation is

$$(21)\qquad S^2 + S\left(\frac{\alpha}{C(R + \alpha r_1)} + \frac{2}{Cr_2}\right) + \frac{1}{C^2 r_2 (R + \alpha r_1)} = 0$$

The character of the roots depends upon the constants of the scheme and in particular upon the three resistances R, r_1 and r_2. If we choose different values for these resistances, we may obtain the different types of singular points investigated earlier. First of all, when $R = 0$, hence $\alpha = 1$, we obtain two real negative roots, i.e. a stable node. This had to be expected because when $R = 0$ the tube I transmits the voltage to the external circuit and consequently does not play any role. In the absence of electronic tubes in a circuit consisting uniquely of capacitance and ohmic resistance, the only possible equilibrium states are stable nodes (only damping of a periodic motion can take place). In order to simplify further we shall assume that r_1 and r_2 are variable and R constant, and for convenience choose R such that $\alpha = -1$. This time the roots are real if

$$(22)\qquad \{r_2^2 + 4(R - r_1)^2 - 8r_2(R - r_1)\} > 0.$$

The boundary between the regions of real and of complex roots is the curve

$$\{(r_2 - 4(R - r_1))^2 - (2\sqrt{3}\,(R - r_1))^2\} = 0$$

The left-hand side is the product of two linear factors. Hence the roots of the characteristic equation are real outside of a wedge in the (r_1,r_2)-plane above the r_1-axis formed by the straight lines:

$$r_2 = (4 \pm 2\sqrt{3})(R - r_1)$$

which cross each other at the point $r_1 = R$, $r_2 = 0$ (Fig. 188). For all points (r_1,r_2) within the wedge, the characteristic roots are complex and the singular point is a focus. On the other hand the real part of the roots is negative whenever

$$r_1 < R, \qquad -r_2 + 2R - 2r_1 > 0.$$

Consequently, the points under the line $r_2 + 2r_1 = 2R$ correspond to stable nodes and foci, while those above the line correspond to unstable nodes and foci. From (21) we see that the product of the roots, which is equal to the last term, has the sign of $+(R + \alpha r_1) = R - r_1$. Hence if the roots are real they have the same sign when and only when $r_1 < R$, i.e. at the left of the vertical $r_1 = R$. At its right the roots are of opposite sign and we have a saddle point. We thus obtain the regions corresponding to the different types of states of equilibrium (Fig. 188). Notice that all the types do occur. The line $r_2 + 2r_1$

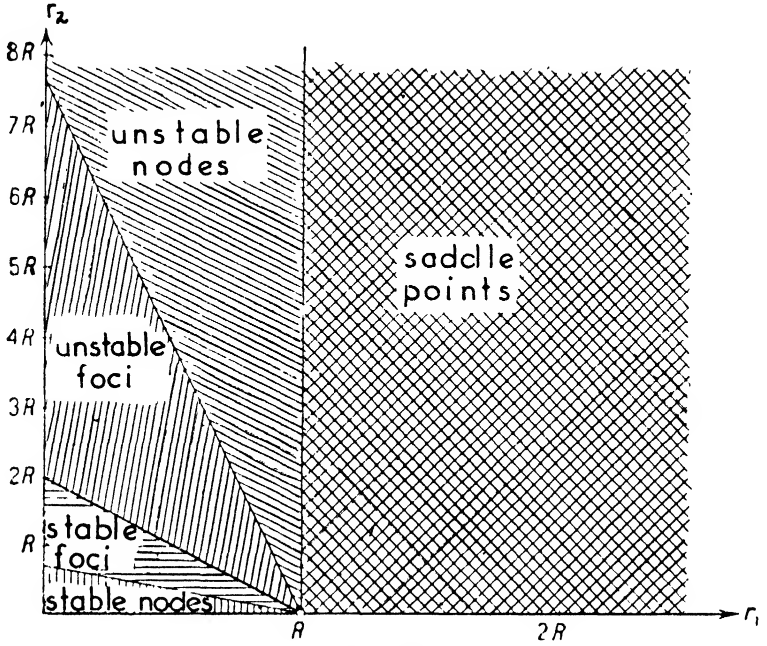


FIG. 188.

$= 2R$ separates the stable and the unstable equilibrium states and consequently defines the conditions of self-excitation of the circuit. If this condition is fulfilled and the singular point is unstable, we may only affirm that the system leaves the equilibrium state and though we can determine the character of this motion, owing to linearity, we cannot say anything concerning the further behavior of the system.

Thus, in the scheme under investigation, if we vary the parameters r_1 and r_2 (or even only r_1 when r_2 has an adequate fixed value) we obtain equilibrium states corresponding to the various types of singular points of first order. This is why we have called the present scheme "universal." It has actually two degrees of freedom, since there always

exist two independent currents.[1] However each of these degrees of freedom is degenerate owing to the absence of inductance, and so the system is described by two equations of the first order, and not of the second order as in the presence of inductance. Hence the system can be studied by the methods developed in this book. Let us note that when $r_2 = 0$ the universal scheme is transformed into a scheme with one degree of freedom described in the preceding chapter (Chap. IV, §10).

§4. States of Equilibrium and Their Stability

We recall the basic definition of stability in the sense of Liapounoff: A state of equilibrium $A(x_0,y_0)$ is stable whenever given a circular region Ω of center A and arbitrary radius ϵ there exists a corresponding circular region Ω' of center A and radius $\delta(\epsilon)$ such that upon following any path issued from any point M of Ω' one never reaches the boundary circumference of Ω. Whenever these conditions do not hold we have instability. The reader with this definition before him will readily verify that in the sequel "stability" and "instability" are used in accordance with this definition, and also that they were so used in §2.

There is a basic theorem of Liapounoff which will enable us to determine the stability properties of the singular points of our system.

The singular point $A(x_0,y_0)$ is an intersection of the curves $P = 0$, $Q = 0$, so that

$$P(x_0,y_0) = Q(x_0,y_0) = 0. \tag{23}$$

To determine the stability of the dynamical system relative to A is to find what happens when the system is disturbed from the equilibrium position represented by A to some position nearby represented by a point $M(x_0 + \xi, y_0 + \eta)$. We may consider ξ,η as new coordinates corresponding to a translation of the origin to A and we will refer for the present to A indifferently as the origin (understood for the coordinates ξ and η) or the position of equilibrium.

Since P,Q are polynomials we may expand $P(x_0 + \xi, y_0 + \eta)$ and $Q(x_0 + \xi, y_0 + \eta)$ in powers of ξ and η. Upon doing so and recalling (23), we obtain by substituting in the basic system (1) a new system

$$\begin{cases} \dot{\xi} = a\xi + b\eta + (p\xi^2 + 2q\xi\eta + r\eta^2 + \cdots), \\ \dot{\eta} = c\xi + d\eta + (p_1\xi^2 + 2q\xi\eta + r_1\eta^2 + \cdots), \end{cases} \tag{24}$$

[1] According to Kirchhoff, the number of degrees of freedom or the number of independent currents in an electric circuit is determined by the minimum number of breaks of the circuit necessary to eliminate all the closed circuits.

$$(25) \qquad \begin{cases} a = P_{x_0}(x_0,y_0), & b = P_{y_0}(x_0,y_0), \\ c = Q_{x_0}(x_0,y_0), & d = Q_{y_0}(x_0,y_0). \end{cases}$$

It might very well happen that the right hand sides in (24) begin with terms of degree $n > 1$. This means that the curves $P = 0$, $Q = 0$ have A for multiple point. We will then say that A is a *singular point of higher type*, and more precisely a *singular point of order n*. The only case discussed here is where $n = 1$ and in fact $ad - bc \neq 0$. The singular point is then said to be *simple*.

If we preserve only the terms of degree unity in (24) there results a linear system, which is much simpler and is known as the system of the first approximation:

$$(26) \qquad \dot{\xi} = a\xi + b\eta, \qquad \dot{\eta} = c\xi + d\eta.$$

Now the right-hand sides of (24) are polynomials in ξ,η whose coefficients are linear combinations of the coefficients of the initial polynomials in x,y. Hence a small change in the new coefficients will result likewise in a small change in the old. From this we conclude (since the paths of (1) and (24) are the same) that the system (24) likewise has stable structure.

The characteristic equation

$$(27) \qquad S^2 - (a + d)S + ad - bc = 0$$

and its roots S_1,S_2 are referred to as characteristic equation and roots of (24) or of (1) for the singular point A. The same argument as for linear systems in §2, will show here also that owing to the stable structure of (24) the roots are neither zero nor pure complex, and that they are distinct. Under these circumstances the discussion of §2 yields the behavior of the paths of the first approximation throughout the whole plane, and in particular in the neighborhood of the singular point A. Now Liapounoff has shown that in substance the same situation holds regarding the system (24). More precisely the behavior of the paths of (1) near the singular point A is the same as for the first approximation. The complete derivation of Liapounoff's results will be found elsewhere. Intuitively however they may be readily apprehended as follows. Introducing polar coordinates ρ,θ by the relations $\xi = \rho \cos \theta$, $\eta = \rho \sin \theta$, the right-hand sides P,Q of (24) become

$$P = \rho\{a \cos \theta + b \sin \theta + \rho(\cdot\ \cdot\ \cdot)\},$$
$$Q = \rho\{c \cos \theta + d \sin \theta + \rho(\cdot\ \cdot\ \cdot)\}.$$

For ρ very small and θ arbitrary the terms $\rho(\cdot\ \cdot\ \cdot)$ in the brackets are

very small with respect to the others and so one may write approximately

$$P \doteq \rho(a \cos \theta + b \sin \theta) = a\xi + b\eta,$$
$$Q \doteq \rho(c \cos \theta + d \sin \theta) = c\xi + d\eta.$$

This amounts to taking P,Q the same as for the first approximation. Hence the paths of the given system and of its first approximation will behave almost alike; their general pattern will be the same, always of course near the singular point A.

We state explicitly the *Theorem of Liapounoff: In a system* (1) *with stable structure the equilibrium position A is stable when and only when the corresponding characteristic roots have negative real parts and the behavior pattern about A as regards nodes, foci or saddle points is the same as for the first approximation.*

In the excluded cases the first approximation is not sufficient to determine the stability properties and an additional investigation is necessary.

We will describe the situation in more detailed manner:

I. *S_1 and S_2 are both real and of the same sign.* If they are both negative the singular point A is a stable node while if they are both positive it is an unstable node.

II. *S_1 and S_2 are real but of opposite sign.* Then A is a saddle point.

III. *S_1 and S_2 are both complex.* They are then conjugate complex say $S_1 = \alpha + \beta i$, $S_2 = \alpha - \beta i$. Their common real part is α. The point A is a stable focus when $\alpha < 0$ and an unstable focus when $\alpha > 0$.

The characteristic equation (27) in the explicit form is

$$S^2 - (P_{x_0} + Q_{y_0})S + (P_{x_0}Q_{y_0} - P_{y_0}Q_{x_0}) = 0. \tag{28}$$

Finally in the case of the saddle point there are four paths tending to the saddle point from two directions of slopes k_1,k_2. Referring to our discussion of linear systems we find that k_1,k_2 are the roots of the equation

$$P_{y_0}k^2 + (P_{x_0} - Q_{y_0})k - Q_{x_0} = 0. \tag{29}$$

§5. Application to a Circuit with Voltaic Arc

To illustrate the method of Liapounoff we shall discuss the equilibrium of the circuit with a voltaic arc connected through an inductance and a shunted capacitance, sketched in Fig. 189. This circuit differs slightly from the circuit of an arc oscillator, but for our present

problem it represents the most general case. The circuits with voltaic arc including either capacitance (Chap. IV, §11) or inductance alone (Chap. IV, §7, No. 1) are obtained as a consequence of "degeneracy" of the general case, i.e. by making one of the two parameters L or $1/C$

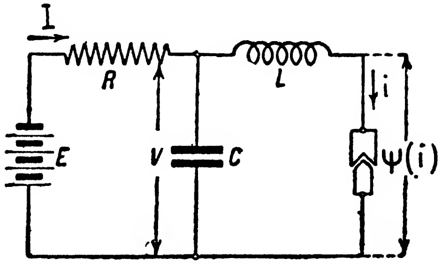


FIG. 189.

equal to zero. In the notations of Fig. 189 Kirchhoff's laws yield the following equations:

$$V = L\dot{i} + \psi(i), \qquad RI + V = E, \qquad I = i + C\dot{V}$$

where $\psi(i)$ is the characteristic of the voltaic arc, i.e. the function expressing the dependence between the voltage on the terminals of the

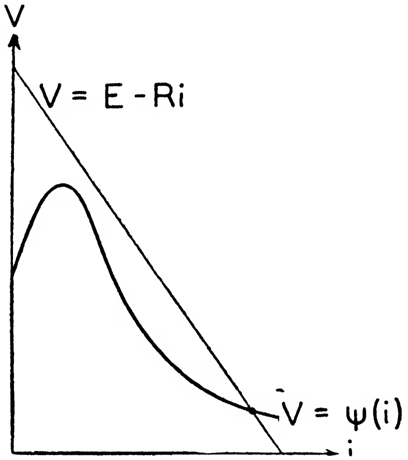


FIG. 190.

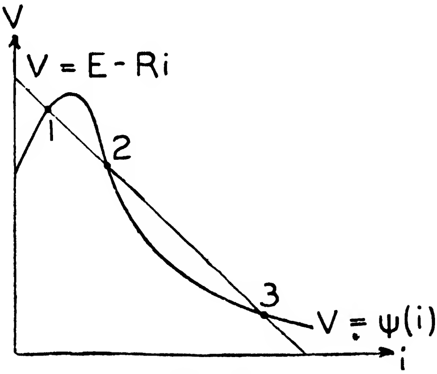


FIG. 191.

arc and the current through the arc (Figs. 190, 191). Eliminating I from the three equations, we obtain two differential equations of the first order:

$$\dot{V} = \frac{E - V - Ri}{RC}, \qquad \dot{i} = \frac{V - \psi(i)}{L} \tag{30}$$

The equilibrium states are defined by $\dot{V} = 0$ and $\dot{i} = 0$, or

$$V = E - Ri, \qquad V = \psi(i). \tag{31}$$

The intersections of the first graph, a straight line, with the second which is the characteristic give the states of equilibrium, of which there may be one or three. Let (V_0, i_0) be one of the intersections, so that

$$V_0 = E - Ri_0, \qquad V_0 = \psi(i_0). \tag{32}$$

Setting as required by the method of Liapounoff, $i = i_0 + \xi$, $V = V_0 + \eta$, then expanding $\psi(i)$ in series we find

$$\psi(i_0 + \xi) = \psi(i_0) + \xi\psi'(i_0) + \cdots .$$

Keeping only the first degree terms and substituting in (30), we have in view of (32), the system of the first approximation

$$\dot{\eta} = -\frac{\eta}{RC} - \frac{\xi}{C}, \qquad \dot{\xi} = \frac{\eta}{L} - \frac{\rho\xi}{L},$$

where $\rho = \psi'(i_0)$ is the slope of the characteristic of the arc, at the point corresponding to the given equilibrium state. Its dimension is that of resistance. The magnitude ρ, the resistance of the arc, is a variable which for certain values of i_0 can become negative. When using this term, however, one must keep in mind all the assumptions made when we introduce the term "negative resistance" (see Chap. I, §6). The characteristic equation is

$$\left(\frac{\rho}{L} + S\right)\left(\frac{1}{RC} + S\right) + \frac{1}{LC} = 0,$$

or in expanded form:

$$S^2 + \left(\frac{1}{RC} + \frac{\rho}{L}\right) S + \frac{1}{LC}\left(\frac{\rho}{R} + 1\right) = 0. \tag{33}$$

The properties of the characteristic roots S_1, S_2 will govern stability. Actually we do not need to solve for the roots and the following elementary information will be sufficient:

$$S_1 + S_2 = -\left(\frac{1}{RC} + \frac{\rho}{L}\right) = \alpha,$$

$$S_1 S_2 = \frac{1}{LC}\left(\frac{\rho}{R} + 1\right) = \beta.$$

From this we conclude, since L, R, C are always positive that:

(a) If S_1 and S_2 are real they have the same sign when $\rho > -R$ and opposite sign when $\rho < -R$. Moreover if they have the same sign, it is the sign of α, and they are both negative when $\rho > -L/RC$, and both positive otherwise.

As we have seen (§2) if the roots are complex S_1S_2 is always positive, and the common real part $\gamma = \frac{1}{2}(S_1 + S_2) = \frac{1}{2}\alpha$. Hence:

(b) If $\rho < -R$, S_1S_2 is negative, hence S_1 and S_2 are real and of opposite sign. The singular point under consideration is then a saddle point, and so it is unstable.

(c) If S_1,S_2 are complex we have stability when and only when $\alpha < 0$, i.e. when and only when $\rho > -L/RC$.

(d) Whenever $\rho > -R$ and $R < \sqrt{L/C}$, the singular point is stable, and certainly:

(e) Whenever $\rho > 0$ we have stability.

Finally the condition for complex roots is

$$\left(\frac{1}{RC} + \frac{\rho}{L}\right)^2 - \frac{4}{LC}\left(\frac{\rho}{R} + 1\right) < 0.$$

The left-hand side is identically

$$\left(\frac{\rho}{L} - \frac{1}{RC}\right)^2 - \frac{4}{LC} = \left(\frac{\rho}{L} - \frac{1}{RC} + \frac{2}{\sqrt{LC}}\right)\left(\frac{\rho}{L} - \frac{1}{RC} - \frac{2}{\sqrt{LC}}\right).$$

The two factors vanish respectively for

$$\frac{\rho}{L} = \frac{1}{RC} - \frac{2}{\sqrt{LC}} = \lambda, \qquad \frac{\rho}{L} = \frac{1}{RC} + \frac{2}{\sqrt{LC}} = \mu.$$

It is clear that $\lambda < \mu$. The product of parentheses can only be negative when ρ/L is between λ and μ or when and only when

$$\text{(34)} \qquad \frac{1}{RC} - \frac{2}{\sqrt{LC}} < \frac{\rho}{L} < \frac{1}{RC} + \frac{2}{\sqrt{LC}}.$$

Recalling now that $\rho = \psi'(i_0) =$ the slope of the characteristic at (i_0, V_0), we see by reference to Fig. 191, that at the point 1 the value of $\rho > 0$ and at the points 2,3 its value < 0. Hence from property (e) the point 1 is always a stable singular point. If ρ is quite large, i.e. if the characteristic is quite steep at the point 1, (34) will not hold, the roots S_1,S_2 will be real and we will have a stable node. On the other hand if for example the inductance L is very large, (34) will hold at the point and it will be a stable focus.

Notice now that if $R < \frac{1}{2}\sqrt{L/C}$ the number $\lambda > 0$, and so at the points 2,3 where $\rho < 0$, (34) does not hold. If we recall that the point 3 always exists and refer to property (d) we have:

(f) Whenever $R < \frac{1}{2}\sqrt{L/C}$, roughly speaking when the resistance is small, there is at least one singular point which is a stable node.

Finally at the point 2 we have $\rho < 0$ (negative slope and $|\rho| > R$). hence by (b):

(g) The point 2 in Fig. 191 always corresponds to a saddle point.

Property (g) contradicts the results obtained in Chap. IV, §11 for a circuit consisting of a voltaic arc and either capacitance or inductance alone. In these schemes the middle equilibrium state was stable when the circuit included capacitance and unstable when it included inductance. When both inductance and capacitance are present, the middle equilibrium state is always a saddle point and is unstable. This apparent contradiction belongs to the range of questions which arise when we pass from a system described by an equation of second order to a degenerate system described by an equation of first order, in particular, when we pass from the system where L and C are different from zero to a system where either L or C is assumed to be zero. These questions and, in particular, the relation between the character of equilibrium in a degenerate system and in the initial system will be discussed in a separate section.

§6. Periodic Motions and Their Stability

Let us examine now the periodic motions which may take place in our basic system (1). If $x(t),y(t)$ is a motion and there exists a positive T (we assume that T is its smallest possible value) such that for all values of t

$$x(t + T) = x(t), \qquad y(t + T) = y(t),$$

the motion is called periodic of period T. We know that to any periodic motion there corresponds a closed path and conversely, to every closed path there corresponds an unlimited number of periodic motions which differ from one another only by the choice of the time origin. We have examined the closed paths in the case of conservative systems where they always made up a whole continuum of concentric ovals (for example, paths around a center). In certain self-oscillatory systems we found an isolated closed path or limit-cycle towards which the neighboring paths spiral on both sides.

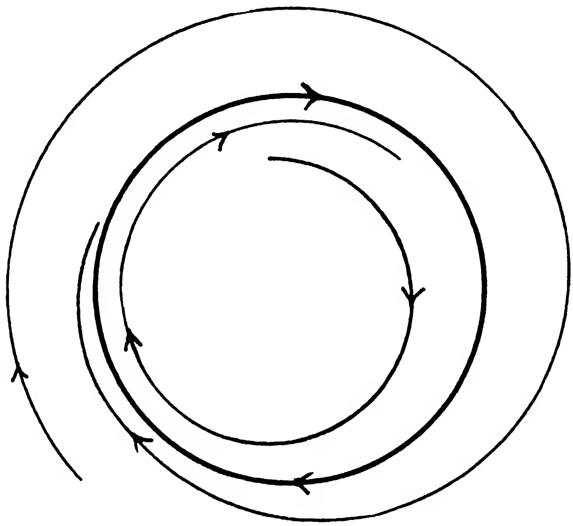
It is not difficult to show that if γ is an isolated closed path then on any one of the two sides of γ all the neighboring paths spiral towards γ,

or else all spiral away from γ as $t \to +\infty$. Hence three types of limit-cycles are possible:

Orbitally stable limit-cycle: on both sides of γ the spiralling is towards γ.

Orbitally unstable limit-cycle: on both sides of γ the spiralling is away from γ.

Orbitally semi-stable: on one side the spiralling is towards γ and on the other away from γ.



stable limit-cycle
FIG. 192.

It may be shown that in *structurally stable systems the semi-stable type does not occur.* If the reader prefers we shall rule out the semi-stable type from our considerations, as it does not occur in the applications.[1] In the sequel we shall only consider orbital stability or instability of limit-cycles and so we shall merely say "stability" or "instability."

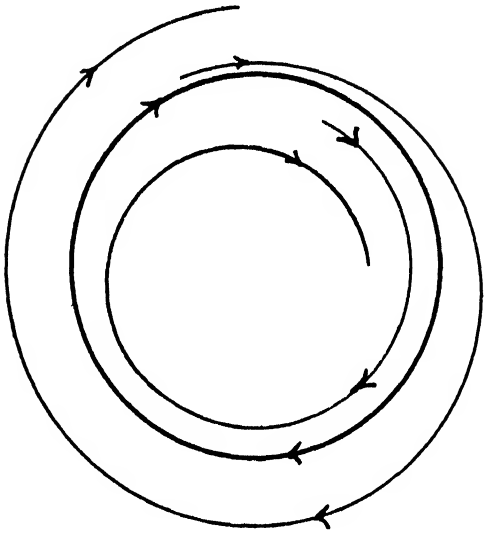
[1] The periodic motion $x = \bar{x}(t)$, $y = \bar{y}(t)$, of period T, is stable in the sense of Liapounoff whenever for every $\epsilon > 0$ there is a $\delta(\epsilon) > 0$ such that if

$$|x(t_0) - \bar{x}(t_0)| < \delta, \qquad |y(t_0) - \bar{y}(t_0)| < \delta$$

then whatever $t > t_0$

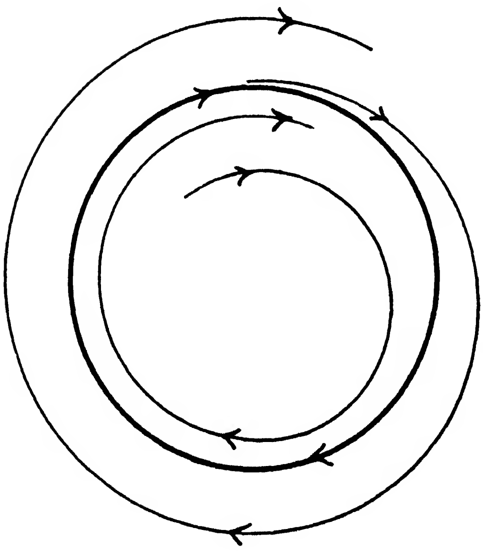
$$|x(t) - \bar{x}(t)| < \epsilon, \qquad |y(t) - \bar{y}(t)| < \epsilon.$$

In other words if at any time the representative points M, M' on the closed path γ and a nearby path γ_1, are sufficiently close together, they remain quite close together ever after.



unstable limit-cycle

Fig. 193.



semi-stable limit-cycle

Fig. 194.

As in the case of singular points the consideration of perturbed motions will lead to an auxiliary linear system. Let a motion very near the periodic one be $x(t) = \bar{x}(t) + \xi(t)$, $y(t) = \bar{y}(t) + \eta(t)$. Substituting in the system (1) we have

$$\begin{aligned} \dot{\xi} &= P(\bar{x} + \xi, \bar{y} + \eta) - \dot{\bar{x}}, \\ \dot{\eta} &= Q(\bar{x} + \xi, \bar{y} + \eta) - \dot{\bar{y}}. \end{aligned}$$

Since P,Q are polynomials the first terms at the right may be expanded in powers of ξ,η. Upon doing so and neglecting the terms in ξ,η of degree > 1, we obtain the so-called equations of variation

$$\begin{aligned} \dot{\xi} &= P_{\bar{x}}(\bar{x},\bar{y})\xi + P_{\bar{y}}(\bar{x},\bar{y})\eta \\ \dot{\eta} &= Q_{\bar{x}}(\bar{x},\bar{y})\xi + Q_{\bar{y}}(\bar{x},\bar{y})\eta \end{aligned} \tag{35}$$

Since $P_{\bar{x}},P_{\bar{y}},Q_{\bar{x}},Q_{\bar{y}}$ are functions of the periodic arguments, $\bar{x},\bar{y}$ having T for period, (35) is a system of linear equations with periodic coefficients with the period T. The general form of the solution of such a system is

$$\begin{aligned} \xi &= C_1 e^{h_1 t} f_{11}(t) + C_2 e^{h_2 t} f_{12}(t), \\ \eta &= C_1 e^{h_1 t} f_{21}(t) + C_2 e^{h_2 t} f_{22}(t), \end{aligned} \tag{36}$$

where the f_{ik} are periodic functions with period T, and the exponents h_1 and h_2 are called "characteristic exponents." The sign of their real part determines whether these solutions are growing or damping down. What we have just said holds for a general system of two linear equations with periodic coefficients, and whose characteristic exponents have non-zero real parts, and in particular are $\neq 0$. It so happens that since the system (1) is autonomous, one of the two exponents is actually zero. Roughly speaking this is due to the fact that essentially one may reduce the number of equations to one. In outline this may be shown as follows. Let γ be the closed path of a periodic motion and δ a path nearby. In a suitably small neighborhood of γ one may determine the position of a point N as follows. Let $n = MN$ be the shortest normal from N to γ and let s be the length of the arc AM from a fixed point A of γ to M measured positively in the direction of the motion along γ. Let also units be so chosen that the length of γ is unity. Thus along γ, $t = f(s)$ where f is monotonic increasing and as s increases by one, t increases by T. As for $\dot{n}$, the component along MN of the vector (P,Q) at N, it depends solely upon M and N. Thus

$$\dot{n} = g(n,s), \qquad t = f(s)$$

where f',g are analytic and of period *one* in s. Hence the basic equation

for n is

$$\frac{dn}{ds} = f'(s)g(n,s) = F(n,s)$$

where F is analytic and of period *one* in s. For $n = 0$, the vector (P,Q) is tangent to γ and so $F(0,s) = 0$. Expanding then $F(n,s)$ in powers of n, and since n is small, neglecting its powers higher than one, we

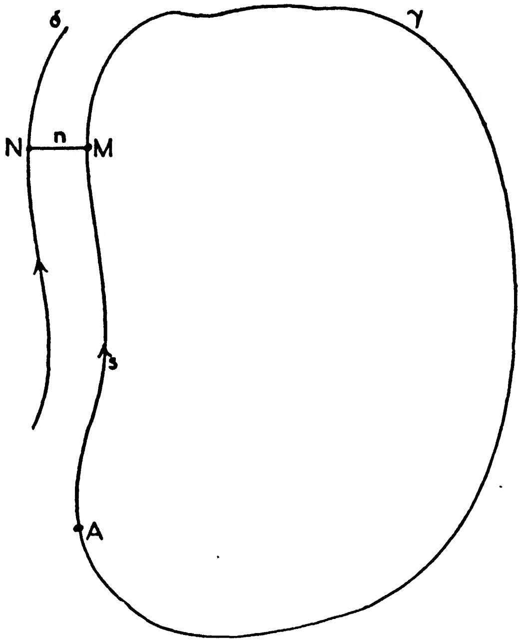


FIG. 195.

obtain an equation of variation relative to the periodic solution $n = 0$ which is

(37) $$\frac{dn}{ds} = F_n(0,s)n = \phi(s)n.$$

Let $\phi(s)$ be developed in Fourier series over the interval unity:

(38) $$\phi(s) = a_0 + \Sigma a_p \cos(2\pi ps + \alpha_p)$$

Then

$$a_0 = \int_0^1 \phi(s)\, ds$$

and it may be shown that

$$a_0 = h = \frac{1}{T}\int_0^T (P_{\bar{x}} + Q_{\bar{y}})\, dt.$$

Now from (37) and (38) follows by integration

$$\log n = C + hs + \sum \frac{a_p \sin(2\pi ps + \alpha_p)}{2p}$$

and therefore

$$n = e^{hs}\psi(s), \tag{39}$$

where ψ is periodic and of period unity. Since s and $t \to +\infty$ simultaneously, $n \to 0$ when and only when $h < 0$. For obvious reasons we shall refer to h as the characteristic exponent of the closed path γ. Thus a n.a.s.c. for the closed path γ to be stable is that its characteristic exponent be negative and a n.a.s.c. for γ to be unstable is that the exponent be positive. When it is zero no conclusion can be drawn from the above considerations regarding stability.

For more details regarding the above geometric argument the reader may be referred to the recent Princeton thesis by Diliberto.

§7. THE LIMITING CONFIGURATIONS OF THE PATHS

Let γ be a path of our system (1) and let γ_M^+ be a positive half-path (see §1) which does not tend to infinity as the representative point follows it indefinitely beyond M. We ask what is the totality Λ^+ of the points P to which γ_M^+ tends, that is the totality of the points P such that with indefinitely increasing time the representative point following γ_M^+ passes infinitely often arbitrarily near P. Similarly for γ_M^- described backwards, the corresponding configuration is written Λ^-. There is a theorem of Poincaré and Bendixson giving a complete answer to the question. Let us call Λ^+ the positive and Λ^- the negative limiting set. In the case under consideration of structurally stable systems it may be shown that when γ is not a closed path the following situation holds.

I. *The positive limiting set Λ^+ can only be a stable focus, a stable node, a saddle point, or a stable limit-cycle. Similarly for Λ^- with "stable" replaced by "unstable."*

II. *The curves tending to saddle points are such that none joins two saddle points. That is to say if γ_M^+ tends to a saddle point γ_M^- does not and conversely.*

From this we deduce the following very useful result due to Poincaré:

III. *Let Ω be a bounded region which is closed (includes its own boundary) and let it be free from singular points. Then if Ω contains a half-path (positive or negative) then Ω contains a limit-cycle.*

For if Ω contains say γ_M^+ then it contains Λ^+ and the latter can only be a limit-cycle.

A noteworthy special case is

IV. *Let Ω be free from singular points and let its boundary consist of several ovals $C_1, \cdots, C_k$ such that along each C_i the velocity vector (P,Q) always points inward. Then Ω contains a limit-cycle. Similarly if the velocity vector always points outward.*

For clearly in the first case if γ ever crosses the boundary when followed forward then it can never return to it and so must remain in Ω. Hence Ω contains some γ_M^+. Similarly in the second case with γ followed backward, Ω now containing some γ_M^-.

The results just discussed enable us to give a very clear meaning to the concept of self-oscillation in systems (1) with stable structure. For the only time there is a self-oscillation is when the system is already on a closed path γ or else tends to such a γ as t increases indefinitely. Moreover in a physical system it is not sufficient that it be *on* γ, it must tend to remain there whenever slightly deviated. That is to say γ must be stable, and be the limit of the neighboring paths as $t \to +\infty$. In other words self-oscillations mean stable limit-cycles. It is important in this connection to state another property of our systems, readily proved but again admitted here:

V. *There is at most a finite number of limit-cycles.* In particular there is at most a finite number of stable limit-cycles. Each of them corresponds to an oscillation of definite amplitude and frequency. For if $x(t),y(t)$ is one of the motions along one of them, say γ, then all the others are of the type $x(t-\tau),y(t-\tau)$ and have the same amplitude and frequency. We add (provable but not proved here) that

VI. *In a system of stable structure the limit-cycles have one non-zero characteristic exponent.*

To sum up then:

VII. *In a system with stable structure there may take place at most a finite number of oscillatory motions, each with a definite amplitude and frequency.*

The role of the unstable limit-cycles is in a sense that of a "repeller." If the representative point starts say inside of one γ, it behaves entirely differently than if it started outside. Thus γ is a "behavior separator" for the paths.

From the practical point of view it is also important to point out the following consequence:

VIII. *In a physical system represented by a system of two differential equations of order unity, the only possible stationary regimes are stable states of equilibrium and stable oscillations.*

§8. THE INDEX IN THE SENSE OF POINCARÉ

This concept is particularly useful in correlating the positions of singular points and limit-cycles. Its chief utilization is often in excluding certain combinations of these elements.

The basic system (1) being the same as before consider in the x,y plane *a simple closed curve N which does not cross equilibrium states.* Take on this curve a point $S(x,y)$ and with this point as origin draw the vector $(P(x,y),Q(x,y))$ which is tangent to the path through S. If we move the point S along the curve N, the vector will rotate. When the point S describes the closed curve N and returns to its

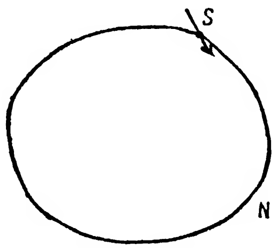


FIG. 196.

initial position, the vector will have experienced a certain number of revolutions, i.e. it will have revolved j times, where j is an integer. We shall consider the direction of rotation as positive when it coincides with the direction of motion of the point around the curve N. To be more definite, we shall say that the point S moves around the curve N always in counterclockwise direction. Thus j can be either a positive or a negative integer or zero. The number j is in a certain sense *independent of the form of the curve N.* In fact, if N changes continuously without crossing singular points, the angle of rotation of the vector can only change continuously; and since it is an integer times 2π, it is constant. Therefore all the closed curves surrounding the same singular points will produce the same number j. The number j is called the *index of the closed curve N with respect to the field of the vectors* (P,Q).

Let us surround an equilibrium state, a singular point, by the small oval N. As we have seen, the index is independent of N if this curve does not surround other singular points. Consequently the index is determined by the character of the singular point. Therefore the index of such a closed curve depends upon the singular point alone and we refer to it as the *index of the singular point* under consideration. A direct examination shows clearly that the Poincaré indices of a

center, a node, and a focus are all $+1$, and the Poincaré index of a saddle point is -1. The validity of the following statements can also be proved by direct examination of the figures:

1. *The index of a closed curve, which does not surround singular points, is equal to zero* (Fig. 198).
2. *The index of a closed curve which surrounds several singular points is equal to the sum of the indices of these points.*
3. *The index of a closed path is* $+1$ (see Fig. 197, case of a center).

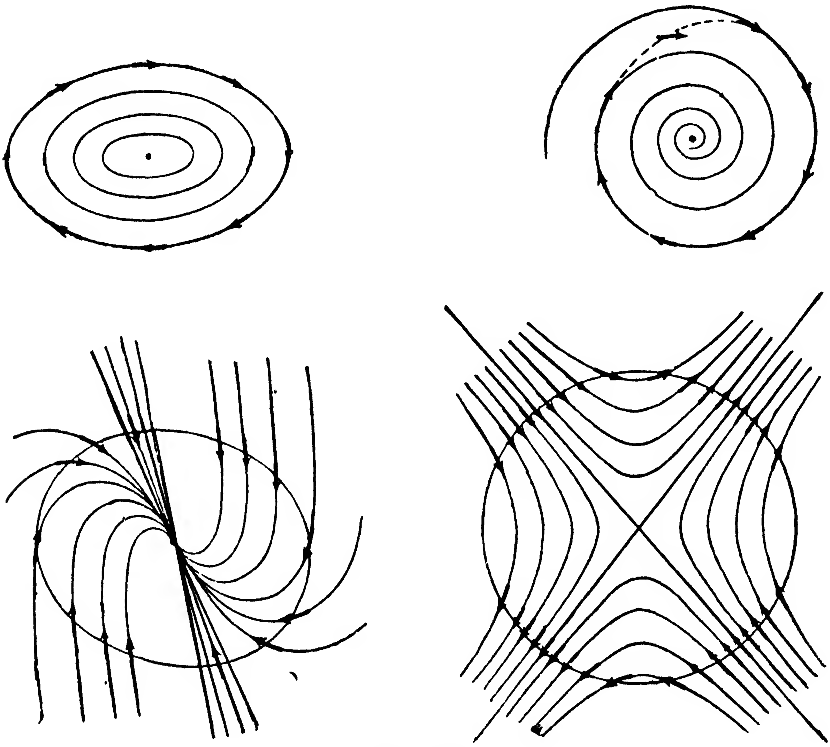


FIG. 197.

4. *The index of a closed curve along which the vectors either all point inwards or all outwards is* $+1$ (see Fig. 197, case of a node).

These statements, which we have obtained by direct investigation of particular examples and by considerations of continuity based upon geometric intuition, can be proved analytically. It is easy to see that the index of the closed curve N can be expressed by the curvilinear integral

$$j = \frac{1}{2\pi} \oint_N d \left\{ \arctan \frac{Q(x,y)}{P(x,y)} \right\} = \frac{1}{2\pi} \oint_N \frac{P\,dQ - Q\,dP}{Q^2 + P^2}.$$

This is a curvilinear integral of a total differential; consequently, if within the region bounded by N the corresponding functions under the integral and their derivatives are continuous, the integral is zero. This gives us an exact proof of our first statement, that the index of a closed curve N which does not surround singular points is zero. For according to our assumption concerning P and Q the continuity of the functions under the integral and of their derivatives can be disturbed only at the points where $P(x,y) = 0$, and $Q(x,y) = 0$, i.e. at the singular

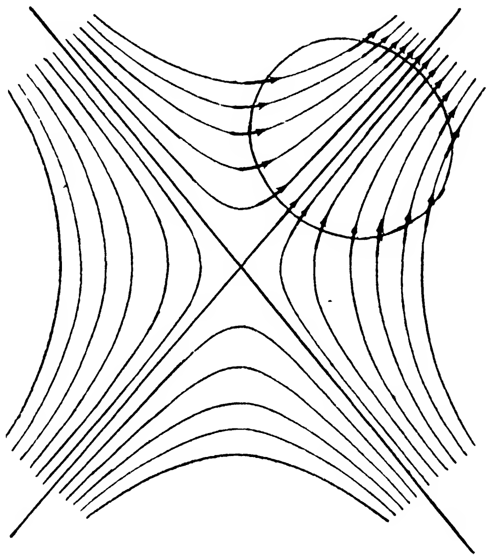


FIG. 198.

points. Let us calculate the index of a singular point. Under our assumptions (stable structural system) we have only ordinary singular points. Taking the one under consideration as the origin we will have

$$P = ax + by + P_2(x,y)$$
$$Q = cx + dy + Q_2(x,y)$$
$$ad - bc \neq 0,$$

where P_2,Q_2 contain only terms of degree at least two.

Let us first prove that in the calculation of the index we can neglect all the terms of high order, i.e. P_2 and Q_2. According to our previous results, the index is independent of the form of the curve and consequently we can identify the curve N with a circle of sufficiently small radius ρ ($\rho > 0$).

Let us pass to polar coordinates $x = \rho \cos \phi$, $y = \rho \sin \phi$, and transform the curvilinear integral into an ordinary definite integral:

$$j = I(\rho) = \frac{1}{2\pi}\int_0^{2\pi} \frac{AdB - BdA + \rho F(\rho,\phi)d\phi}{A^2 + B^2 + \rho G(\rho,\phi)},$$

$$A = a\cos\phi + b\sin\phi, \qquad B = c\cos\phi + d\sin\phi$$

where $F(\rho,\phi)$, $G(\rho,\phi)$ are polynomials in ρ with coefficients periodic functions of ϕ. Observe that the definite integral $I(\rho)$ is a continuous function of ρ for sufficiently small ρ. Therefore $\lim_{\rho\to 0} I(\rho) = I(0)$. On the other hand, we know that the curvilinear integral is independent of ρ for sufficiently small ρ. Therefore, when ρ is sufficiently small, $I(\rho) = I(0)$ and finally

$$j = I(0) = \frac{1}{2\pi}\int_0^{2\pi} \frac{AdB - BdA}{A^2 + B^2}$$

Thus we have shown that in the calculation of the Poincaré index of the singular point we can neglect the non-linear terms, that is to say we may substitute for (1) its first approximation. In order to calculate $I(0)$ it is convenient to apply the following method. Let us return to ordinary coordinates and write our expression again in the form of a curvilinear integral:

$$j = I(0) = \frac{1}{2\pi}\oint_N \frac{(ax + by)d(cx + dy) - (cx + dy)d(ax + by)}{(ax + by)^2 + (cx + dy)^2}$$

where N is now any oval surrounding the origin, for the origin is the only singular point of the first approximation. Take in particular as oval the ellipse Γ

$$(ax + by)^2 + (cx + dy)^2 = 1.$$

Then, simple considerations show that

$$j = I(0) = \frac{q}{2\pi}\oint_\Gamma (x\,dy - y\,dx), \qquad q = ad - bc.$$

Let S be the area of the ellipse. By well known results

$$S = \tfrac{1}{2}\oint_\Gamma (x\,dy - y\,dx) = \frac{\pi}{|q|}$$

and therefore $j = q/|q|$. Now $q = S_1S_2 =$ the product of the characteristic roots for our singular point. Therefore

$$j = \frac{S_1S_2}{|S_1S_2|} = \begin{cases} +1 \text{ for a node or focus} \\ -1 \text{ for a saddle point} \end{cases}$$

as asserted before.

We now pass to some corollaries resulting from the theory of indices concerning the coexistence of closed paths and various equilibrium states.

Corollary I. A closed path γ surrounds at least one singular point.

For the index of γ is $+1$ and that of an oval not surrounding singular points is zero.

Corollary II. If a closed path surrounds a single singular point, it can only be a node or a focus.

Corollary III. A closed path surrounds an odd number of singular points and if there are n nodes and foci and s saddle points then $n = s + 1$.

CHAPTER VI

Dynamical Systems Represented by Two Differential Equations of the First Order (Continued)

§1. EFFECT OF THE VARIATION OF A PARAMETER UPON THE PHASE PORTRAIT

We continue to investigate the same general system

$$\dot{x} = P(x,y), \qquad \dot{y} = Q(x,y) \tag{1}$$

as in the previous chapter and until further notice the assumptions made there remain the same: P and Q are polynomials and the system is structurally stable. Small changes in the coefficients do not basically affect the phase portrait.

In the present section we shall investigate what happens when the system depends upon a certain parameter λ, and in particular to what extent the changes in λ affect the phase portrait. In a physical system λ might be an inductance, a resistance, a spring constant, etc. To simplify matters we assume for the system the form

$$\dot{x} = P(x,y,\lambda), \qquad \dot{y} = Q(x,y,\lambda) \tag{2}$$

with P,Q polynomials in all three variables. Our problem may be described more fully in the following way. Suppose that to a given value of λ there corresponds a definite motion and that the representative point M describes a definite path. Let λ vary during a certain period of time and then remain constant. At the end of the process we will have a new phase portrait. Which of the "new" paths will our representative point follow? For a structurally stable system, the answer can be given immediately, provided that the system remains structurally stable during the variation of λ; the representative point always remains in the vicinity of the same attracting element (stable singular point or limit-cycle). If the initial and the final values of λ correspond to a structurally stable system, but the system does not remain structurally stable for all the intermediary values of λ, then, generally speaking, the final motion of the representative point cannot be known. But even in this case it is sometimes possible to settle the question after having studied the variation of the corresponding region

of stability in the large. The cases of soft and hard creation of oscillations (see Chap. IX) where the variation of the phase portrait corresponds to the passage of the representative point from one attracting element to another are examples of this nature.

1. Existence theorem. A simple and well known device is available for examining the solutions as functions of λ. Let us suppose that we wish to investigate the behavior of the solution $x(t),y(t)$ of (2) such that for $t = t_0$ and $\lambda = \lambda_0$ we have $x(t_0) = x_0$, $y(t_0) = y_0$. By adjoining to (2) the auxiliary equation

$$\dot{\lambda} = 0 \tag{2'}$$

we have in (2),(2′) a system of three equations in three unknowns of the same general type as (1), or more generally with right-hand sides analytic in x,y,λ about x_0,y_0,λ_0. This is sufficient to enable us to assert that the solution of (2) which reduces to $\bar{x},\bar{y}$ for $\lambda = \bar{\lambda}$ and $t = t_0$, is analytic in $\bar{x},\bar{y},\bar{\lambda}$ in a certain vicinity of (x_0,y_0,λ_0). For instance if one thinks of $\bar{x},\bar{y},\bar{\lambda}$ as coordinates of a point M in a three space then the solution will be analytic in a certain sphere of center M.

Usually one will think of x_0,y_0 as variable, identify $\bar{\lambda}$ with λ, and view x,y as analytic in x_0,y_0,λ in a certain neighborhood. In point of fact it will generally happen that x_0,y_0 are unrestricted and λ is in a certain interval $\lambda_1 < \lambda < \lambda_2$.

Since the system is autonomous, the general solution is of the form $x(t - t_0,x_0,y_0,\lambda),y(t - t_0,x_0,y_0,\lambda)$ and is the one taking the values x_0,y_0 for $t = t_0$ and λ as specified.

An important complement is the following: Given a time interval T and the solution just written, we can choose x_0',y_0',λ' so close to x_0,y_0,λ that the resulting new solution remains arbitrarily close to the given one in the whole interval from t_0 to $t_0 + T$. In particular small variations of λ, do not, throughout a finite time T, cause very large deviations in a path.

Regarding the phase portrait it is quite another matter. We assume in substance that if λ_0 is any fixed value of λ within the interval considered, then in some small interval $\lambda' < \lambda_0 < \lambda''$ and for any $\lambda \neq \lambda_0$ the system has stable structure. There are now two possibilities regarding λ_0 itself:

I. *The system still has a stable structure for* $\lambda = \lambda_0$. It may then be shown that the portrait of the paths is fixed throughout an interval containing λ_0. In fact the stability of structure is equivalent to the following two properties: (a) For any singular point the corresponding characteristic roots S_1,S_2 are $\neq 0$, not pure imaginary and distinct; (b)

for any closed path γ the corresponding path exponent $h \neq 0$. In the present instance S_1, S_2, h are analytic functions of λ and if they satisfy properties (a), (b) for $\lambda = \lambda_0$, they do so for some interval containing λ_0.

II. *The system fails to have a stable structure for* $\lambda = \lambda_0$. Then λ_0 is a branch point. When λ becomes equal to λ_0 one of a number of things may happen and more particularly: (a) a coincidence of several singular points may take place giving rise to a singular point of higher type; (b) two or more limit-cycles may coincide and disappear as λ crosses λ_0 forward or backward.

2. Influence of the parameter on the nature of the singular points. Let us suppose that for $\lambda = \lambda_0$ there is a singular point at A. Consider a small oval δ surrounding A but neither surrounding nor crossing other singular points occurring for $\lambda = \lambda_0$. Under these conditions, in the vicinity of λ_0, the index

$$j(\lambda) = \frac{1}{2\pi} \int_\delta \frac{P\,dQ - Q\,dP}{P^2 + Q^2}$$

is an analytic function of λ. Since $j(\lambda)$ is an integer it remains constant in a certain interval containing λ_0. Since near λ_0 there are only ordinary singular points, a node or a focus can only merge with a saddle point if they are to disappear as λ crosses λ_0. At all events whatever merging occurs when λ crosses λ_0 the sum of the indices of the points within δ cannot change.

§2. Appearance of a Limit-cycle around a Focus

This case is discussed at length because there is available for it a good "polar coordinate" technique, utilized already by Poincaré and Liapounoff.

It will be recalled that when the origin is a focus then the axes may be so chosen that the system assumes the form

$$\dot{x} = ax - by + P_2(x,y), \qquad \dot{y} = bx + ay + Q_2(x,y)$$

where the polynomials P_2, Q_2 contain only terms of degree two or more. The characteristic roots S_1, S_2 satisfy an equation

$$S^2 + pS + q = 0$$

and here they are both complex and given by

$$S_1 = a + ib, \qquad S_2 = a - ib$$

where

$$S_1 + S_2 = 2a = -p,$$
$$S_1 - S_2 = 2ib = i\sqrt{4q - p^2}, \qquad 4q > p^2,$$
$$2b = +\sqrt{4q - p^2} > 0.$$

We will now make the hypothesis that a,b and the coefficients of P_2,Q_2 are analytic functions of a certain parameter λ and holomorphic for $\lambda = \lambda_0$, and furthermore that $b(\lambda_0) \neq 0$ and hence $b(\lambda_0) > 0$. The value λ_0 will be a branch point, i.e. a place where changes in the portrait of the paths may occur. Since the phase portrait can only change here through S_1,S_2 becoming pure imaginary, we must assume $a(\lambda_0) = 0$. We are particularly interested in the possible appearance or disappearance of a limit-cycle surrounding the focus and tending to it as $\lambda \to \lambda_0$.

The exact form of the system is now

$$(3) \qquad \begin{cases} \dot{x} = a(\lambda)x - b(\lambda)y + P_2(x,y,\lambda) \\ \dot{y} = b(\lambda)x + a(\lambda)y + Q_2(x,y,\lambda) \end{cases}$$

and we write explicitly

$$S_1 = a(\lambda) + ib(\lambda), \qquad S_2 = a(\lambda) - ib(\lambda), \qquad b(\lambda_0) > 0.$$

One will recall that the focus is stable whenever $a(\lambda) < 0$ and unstable whenever $a(\lambda) > 0$.

Passing to polar coordinates r,θ given by $x = r\cos\theta$, $y = r\sin\theta$, we will have

$$r^2 = x^2 + y^2, \qquad \theta = \arctan\frac{y}{x}$$
$$\frac{1}{2}\frac{d(r^2)}{dt} = x\dot{x} + y\dot{y}, \qquad \dot{\theta} = \frac{x\dot{y} - y\dot{x}}{r^2}$$

and hence explicitly

$$(4) \qquad \begin{cases} \dfrac{1}{2}\dfrac{d(r^2)}{dt} = a(\lambda)(x^2 + y^2) + P_2(x,y,\lambda)x + Q_2(x,y,\lambda)y \\ \dot{\theta} = \dfrac{1}{r^2}\{b(\lambda)(x^2 + y^2) + Q_2(x,y,\lambda)x - P_2(x,y,\lambda)y\}. \end{cases}$$

From this follows:

$$\frac{dr}{d\theta} = r\frac{a(\lambda)r + P_2\cos\theta + Q_2\sin\theta}{b(\lambda)r + Q_2\cos\theta - P_2\sin\theta}$$

where in P_2,Q_2 one must write $r \cos \theta$, $r \sin \theta$ for x,y. Now we may write

$$\frac{dr}{d\theta} = \left\{\frac{a}{b} r + \frac{P_2 \cos \theta + Q_2 \sin \theta}{b}\right\} \left\{1 - \frac{P_2 \sin \theta - Q_2 \cos \theta}{br}\right\}^{-1}.$$

Let us set

$$\Omega = \frac{P_2 \sin \theta - Q_2 \cos \theta}{b(\lambda) \cdot r}$$

so that the second bracket is $(1 - \Omega)^{-1}$. We wish to examine the possibility of writing down the well known expansion

$$(5) \qquad (1 - \Omega)^{-1} = 1 + \Omega + \Omega^2 + \cdots$$

From elementary calculus this expansion is known to be valid for $|\Omega| < 1$, and certainly for Ω small. We are only interested in what happens near the focal point, i.e. for r small, and for λ near λ_0, i.e. for $|\lambda - \lambda_0|$ small. Now

(a) Since $b(\lambda)$ is continuous and $\neq 0$ for $\lambda = \lambda_0$, it remains "away from zero" for λ near enough to λ_0. In precise mathematical language, there exist $\alpha,\epsilon > 0$ such that if $|\lambda - \lambda_0| < \epsilon$ then $b(\lambda) > \alpha$.

(b) It will be remembered that P_2,Q_2 are polynomials of degree $\geqslant 2$ in r, hence

$$S(r,\lambda,\theta) = \frac{P_2 \sin \theta - Q_2 \cos \theta}{r}$$

is a polynomial of degree $\geqslant 1$ in r. Its coefficients are analytic functions of λ in the neighborhood of λ_0 ,and polynomials in $\sin \theta$, $\cos \theta$. Hence for ϵ sufficiently small, with $|\lambda - \lambda_0| < \epsilon$, and whatever θ, S tends uniformly to zero when $r \to 0$. That is to say we may choose an $\eta > 0$ such that with $|\lambda - \lambda_0| < \epsilon$, and $r < \eta$, then whatever θ, $|S| < \frac{1}{2}\alpha$.

Under the double assumption then of r and $\lambda - \lambda_0$ sufficiently small, we will have $|\Omega| = \left|\frac{S}{b}\right| < \frac{1}{2}$ and the expansion (5) will be applicable. Under the circumstances we will then have

$$(6) \quad \frac{dr}{d\theta} = \left\{\frac{a(\lambda)r}{b(\lambda)} + \frac{P_2 \cos \theta + Q_2 \sin \theta}{b(\lambda)}\right\} \cdot \left\{1 + \frac{P_2 \sin \theta - Q_2 \cos \theta}{b(\lambda)r} + \left(\frac{P_2 \sin \theta - Q_2 \cos \theta}{b(\lambda)r}\right)^2 \cdots\right\}$$

where the powers are those of Ω. Since Ω, like S is a polynomial in r,

and of degree at least one in r, we may expand the right-hand side in powers of r and so write

$$\frac{dr}{d\theta} = rR_1(\theta,\lambda) + r^2R_2(\theta,\lambda) + r^3R_3(\theta,\lambda) + \cdots \tag{7}$$

where the series converges uniformly for $|\lambda - \lambda_0| < \epsilon$ and $r < \rho$, and this whatever θ. The multiplication shows that

$$R_1(\theta,\lambda) = \frac{a(\lambda)}{b(\lambda)},$$

and that $R_i(\theta,\lambda)$, $i > 1$, is a polynomial in $\sin\theta$, $\cos\theta$ and hence is periodic and of period 2π in θ.

Let $r = f(\theta,r_0,\lambda)$ be the solution of (7) such that $r = r_0$ for $\theta = 0$; i.e. such that

$$r_0 = f(0,r_0,\lambda). \tag{8}$$

It follows from a general theorem due to Poincaré that $f(\theta,r_0,\lambda)$ may be expanded in a power series in r_0 convergent for $\lambda - \lambda_0$ small enough and for all $0 \leqslant \theta \leqslant 2\pi$, provided that $|r_0|$ is sufficiently small: $0 < r_0 < \rho_0$. Moreover since $r = 0$ is a solution of (7) we must have $f \equiv 0$ for $r_0 = 0$, and so the series will have no constant term. Thus

$$r = f(\theta,r_0,\lambda) = r_0u_1(\theta,\lambda) + r_0^2u_2(\theta,\lambda) + r_0^3u_3(\theta,\lambda) + \cdots \tag{9}$$

The coefficients u_k are determined by substituting in (7) and identifying the powers of r_0. Thus we find

$$\begin{cases} \dfrac{du_1}{d\theta} = u_1R_1(\theta,\lambda), \\ \dfrac{du_2}{d\theta} = u_2R_1(\theta,\lambda) + u_1^2R_2(\theta,\lambda) \\ \cdots\cdots\cdots\cdots\cdots \end{cases} \tag{10}$$

In view of the relation (8) we have

$$u_1(0,\lambda) = 1; \qquad u_k(0,\lambda) = 0, \qquad (k = 2,3 \cdots). \tag{11}$$

The differential equations (10) together with the initial conditions (11) enable us to determine $u_1,u_2, \cdots$, in succession. In particular we find

$$u_1(\theta,\lambda) = \exp\left(\frac{a(\lambda)}{b(\lambda)}\,\theta\right),$$

where, as frequently done, we have written for convenience exp (z) for e^z.

We shall now search for arbitrarily small limit-cycles around the focus at the origin. More precisely we are looking for a limit-cycle γ, existing for λ very near λ_0, surrounding the focus, and tending to that focus as $\lambda \to \lambda_0$. Let the representative point M follow such a limit-cycle γ, if any exists from its intersection M_0 with $\theta = 0$, the positive x-axis. Going back to the second equation (4), xQ_2 and yP_2 when expressed in polar coordinates are of degree $\geqslant 3$ in r. Hence this equation may be written

$$\dot{\theta} = b(\lambda) + r(\cdot\cdot\cdot) \doteq b(\lambda)$$

for r and $|\lambda - \lambda_0|$ small enough. Since $b(\lambda_0) > 0$ under the circumstances $\dot{\theta} > 0$, and so as M describes γ the radius vector OM will revolve in the positive (counterclockwise) direction. Thus as θ varies from zero to 2π, M will describe, revolving in the same direction an arc from M_0 to a new point M_1 on the positive x axis. An elementary sketch will then show that γ is a limit-cycle when and only when $M_1 = M_0$, i.e. when and only when $OM_1 = r(2\pi) = OM_0 = r(0) = r_0$. We are thus led to this result: The n.a.s.c. for γ to be a limit-cycle is that $r(2\pi) - r_0 = 0$. That is to say we must have

$$r_0\phi(\lambda,r_0) = f(r_0,2\pi,\lambda) - r_0 = \alpha_1(\lambda)r_0 + \alpha_2(\lambda)r_0^2 + \cdot\cdot\cdot = 0$$

where clearly

$$\alpha_1(\lambda) = u_1(2\pi,\lambda) - 1 = \exp\left(2\pi\frac{a(\lambda)}{b(\lambda)}\right) - 1$$

$$\alpha_k = u_k(2\pi,\lambda), \qquad (k = 2,3, \cdot\cdot\cdot)$$

and also

$$\phi(\lambda,r_0) = \alpha_1(\lambda) + \alpha_2(\lambda)r_0 + \alpha_3(\lambda)r_0^2 + \cdot\cdot\cdot . \tag{12}$$

Thus the basic equation for the existence of the small limit-cycle cutting the x axis at the point $M_0(r_0,0)$ is

$$\phi(\lambda,r_0) = \alpha_1(\lambda) + \alpha_2(\lambda)r_0 + \alpha_3(\lambda)r_0^2 + \cdot\cdot\cdot = 0.$$

The question is whether there is, for $|\lambda - \lambda_0|$ very small, a solution with r_0 very small also. Let us think of λ,r_0 as cartesian coordinates, and for the sake of definiteness suppose $\lambda_0 \geqslant 0$. If this does not hold we could always replace the parameter λ by $\mu = -\lambda$ and we would have $\mu_0 = -\lambda_0 > 0$. Since only $r_0 > 0$ is of interest, we are mainly concerned with the first quadrant. Our inquiry comes down to this:

Does the curve $\phi(\lambda,r_0) = 0$ (we shall merely call it the curve ϕ) possess an arc in the first quadrant ending at the point $A(\lambda_0,0)$?

Since $a(\lambda_0) = 0$ we also have $\alpha_1(\lambda_0) = 0$ and so at all events A is a point of the curve ϕ. We will then have to discuss the behavior of the curve at A. In particular does it have one branch through A (A is then a simple point of ϕ) or several branches through A (A is then a multiple point). In order not to go too far afield we shall only consider the simplest situation:

$$a'(\lambda_0) \neq 0, \qquad \alpha_3(\lambda_0) \neq 0.$$

Notice that

$$\begin{aligned}\alpha_1'(\lambda) &= 2\pi \exp\left(2\pi \frac{a(\lambda)}{b(\lambda)}\right) \cdot \left(\frac{a(\lambda)}{b(\lambda)}\right)' \\ &= 2\pi \exp\left(2\pi \frac{a(\lambda)}{b(\lambda)}\right) \cdot \frac{b(\lambda)a'(\lambda) - b'(\lambda)a(\lambda)}{b(\lambda)^2}\end{aligned}$$

and therefore

$$\alpha_1'(\lambda_0) = 2\pi \frac{a'(\lambda_0)}{b(\lambda_0)} \neq 0.$$

Let us also show that $a(\lambda_0) = 0$ implies $\alpha_2(\lambda_0) = 0$. Indeed to begin with

$$R_1(\theta,\lambda_0) = \frac{a(\lambda_0)}{b(\lambda_0)} = 0.$$

Hence from the first equation (10) we have $\dfrac{du_1(\theta,\lambda_0)}{d\theta} = 0$, which together with the initial conditions yields $u_1(\theta,\lambda_0) = 1$. As a consequence the second equation (10) becomes

$$\frac{du_2(\theta,\lambda_0)}{d\theta} = R_2(\theta,\lambda_0).$$

Since $u_2(0,\lambda_0) = 0$ we have

$$\alpha_2(\lambda_0) = u_2(2\pi,\lambda_0) = \int_0^{2\pi} R_2(\theta,\lambda_0)\, d\theta = 0,$$

for $R_2(\theta,\lambda_0)$ is readily verified to be a homogeneous polynomial of degree three in $\sin\theta$, $\cos\theta$ and hence has a zero integral over the period 2π. More generally it may be shown that the first non-vanishing term for $\lambda = \lambda_0$ in (12) must be odd.

In order that the curve ϕ possess a multiple point at A it is necessary that both partials of the coordinates vanish at the point. Now at

A we have $\phi_{r_0} = \alpha_2(\lambda_0) = 0$, but $\phi_\lambda = \alpha_1'(\lambda_0) \neq 0$. Hence A is an ordinary point.

The slope of the tangent at any point of the curve ϕ is the value of $-(\phi_\lambda/\phi_{r_0})$ at the point. At the point A this expression is infinite. Hence the tangent at A is vertical. To find out whether the arc coming to A rises to the right or to the left of the tangent, we must examine A for a maximum or minimum of λ as a function of r_0, and this requires the knowledge of the second derivative of λ as to r_0 at the point. We have

$$\frac{d\lambda}{dr_0} = -\frac{\phi_{r_0}}{\phi_\lambda}$$

$$\frac{d^2\lambda}{dr_0^2} = -\frac{\partial}{\partial\lambda}\left(\frac{\phi_{r_0}}{\phi_\lambda}\right)\cdot\frac{d\lambda}{dr_0} - \frac{\partial}{\partial r_0}\left(\frac{\phi_{r_0}}{\phi_\lambda}\right).$$

An elementary computation yields

$$\left(\frac{d^2\lambda}{dr_0^2}\right)_A = \frac{-2\alpha_3(\lambda_0)}{\alpha_1'(\lambda_0)} = \frac{-b(\lambda_0)\alpha_3(\lambda_0)}{\pi a'(\lambda_0)}$$

and the sign of this second derivative at A is that of $-\alpha_3(\lambda_0)/a'(\lambda_0)$. We have thus four possibilities.

(a) $a'(\lambda_0) > 0$, $\alpha_3(\lambda_0) < 0$. As λ increases through λ_0, $a(\lambda)$ passes through zero going from minus to plus. Hence the characteristic roots S_1, S_2 have negative real parts for $\lambda < \lambda_0$ and positive real parts for $\lambda > \lambda_0$. Therefore the focus at the origin is stable for $\lambda < \lambda_0$ and unstable for $\lambda > \lambda_0$. Since $(d^2\lambda/dr_0^2)_A > 0$, λ has a minimum at A, and the arc of ϕ comes to A from the right as in Fig. 199. For any particular value λ_1 of λ there is a small limit-cycle when and only when the vertical $\lambda = \lambda_1$ (the dotted vertical in the figures) meets the curve ϕ, i.e. when and only when $\lambda > \lambda_0$. Thus as λ increases and passes λ_0, the focus, stable before λ_0, becomes unstable when λ_0 is passed. Beyond λ_0 however there appears a small limit-cycle γ surrounding the unstable focus and it must be stable since the paths spiralling away from the focus can only tend to γ. The limit-cycle γ tends to the focus as $\lambda \to \lambda_0$.

(b) $a'(\lambda_0) > 0$, $\alpha_3(\lambda_0) > 0$. The discussion is the same save that the point A is now a maximum. The stability of the focus is as before but the new limit-cycle is unstable. As λ increases through λ_0 we have in succession a stable focus and an unstable limit-cycle which tends to the focus as $\lambda \to \lambda_0$. After λ_0 the limit-cycle disappears and the focus becomes unstable (Fig. 200).

(c) $a'(\lambda_0) < 0$, $\alpha_3(\lambda_0) > 0$. The situation is the same as under (a) and Fig. 199, but with stability and instability reversed. As λ increases through λ_0 the focus at first unstable becomes stable beyond λ_0. The new limit-cycle appearing when λ_0 is passed, is unstable.

(d) $a'(\lambda_0) < 0$, $\alpha_3(\lambda_0) < 0$. The situation is the same as under (b) with Fig. 200, and again stability and instability reversed. As λ increases through λ_0 the focus at first unstable becomes stable beyond λ_0. A small stable limit-cycle tends to the focus as λ increasing reaches λ_0.

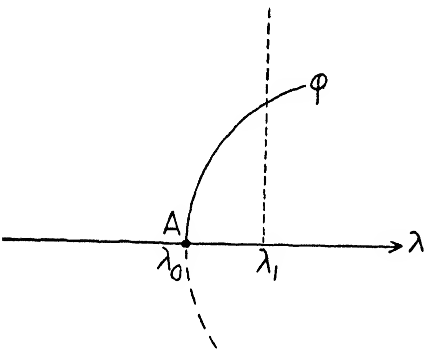


FIG. 199.

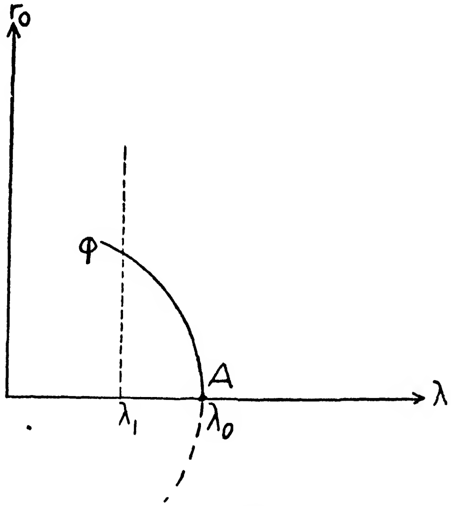


FIG. 200.

1. Application to a certain oscillator. Let us apply the preceding considerations to the self-oscillations of a tuned-grid oscillator when the feed back coupling is increased. The oscillator is the one fully described in Chapter IX, §2 (diagram of Fig. 284), and its equation in the notations there given is:

$$\frac{d^2v}{d\tau^2} + v = (\alpha' + 2\beta' v - 3\gamma' v^2)\frac{dv}{d\tau}$$

where

$$\tau = \omega_0 t, \qquad \alpha' = \omega_0(M\bar{\alpha}_1 - RC), \qquad \beta' = \bar{\beta}_1 M\omega_0, \qquad \gamma' = \bar{\gamma}_1 M\omega_0.$$

Observe that this equation is based upon the usual idealization and that the characteristic of the tube is expressed by a polynomial of the third order: $I_a = V_s(\bar{\alpha}_1 v + \bar{\beta}_1 v^2 - \bar{\gamma}_1 v^3)$ where I_a is the plate current, V_s the saturation voltage, $v = V_g/V_s$, the grid voltage V_g is equal to

the voltage across the condenser. We shall assume that both $\bar{\alpha}_1$ and $\bar{\gamma}_1$ are positive. Departing from our customary notations and to obtain at once a system such as (2) let us set $v = y$, $\dot{v} = x$, and write the differential equation of the second order in the form of a system of two equations of the first order:

$$\dot{x} = -y + (\alpha' + 2\beta' y - 3\gamma' y^2)x, \quad \dot{y} = x.$$

The state $x = y = 0$ is a focus for $|\alpha'| < 2$ and we investigate the possible appearance of a limit-cycle from this focus when the parameter M varies. The characteristic equation is

$$S^2 - \alpha' S + 1 = 0,$$

so that

$$a(M) = \frac{\alpha'}{2} = \frac{\omega_0}{2}(M\bar{\alpha}_1 - RC),$$

$$b(M) = +\sqrt{1 - \frac{\alpha'^2}{4}} = \sqrt{1 - \frac{\omega_0^2}{4}(M\bar{\alpha}_1 - RC)^2}.$$

The branch point M_0 of the parameter M is the value annulling a or $M_0 = RC/\bar{\alpha}_1$. It is the analogue of the λ_0 of the general treatment. We have here also

$$a'(M_0) = \frac{\omega_0 \bar{\alpha}_1}{2}.$$

Let us calculate $\alpha_3(M_0)$. On the basis of (6) and (3) we find

$$R_2(\theta, M_0) = 2\beta_0' \cos^2\theta \sin\theta,$$
$$R_3(\theta, M_0) = -3\gamma_0' \sin^2\theta \cos^2\theta - 4\beta_0'^2 \cos^3\theta \sin^3\theta.$$

Then by (10) for $M = M_0$ we obtain

$$u_2(\theta, M_0) = \tfrac{2}{3}\beta_0'(1 - \cos^3\theta),$$

$$u_3(2\pi, M_0) = \alpha_3(M_0) = -\frac{3}{4}\gamma_0'\pi = -\frac{3\pi\omega_0}{4} RC \frac{\bar{\gamma}_1}{\bar{\alpha}_1}.$$

We have assumed that $\bar{\alpha}_1 > 0$, $\bar{\gamma}_1 > 0$, so that $a'(M_0) > 0$, $\alpha_3(M_0) < 0$. Thus, according to our classification we have the first case (a) when M increases crossing M_0 the focus which is initially stable becomes unstable and a stable limiting·cycle appears. Consequently, when M is sufficiently close to but greater than M_0, the system can undergo a stable self-oscillating process. Let us note that if we had $\bar{\alpha}_1 < 0$, we would be in presence of the second case (b): in this case, when M

increases, the stable focus becomes unstable and the unstable limiting cycle shrinks to a point. It is easy to see that from the physical point of view this last case corresponds to the so-called hard excitation of oscillations, while the first case corresponds to the so-called soft excitation of oscillations. We shall make two more remarks:

(1) If we had taken into account the higher terms of the development of the characteristic, proportional to v^4,v^5,v^6, etc., a reference to

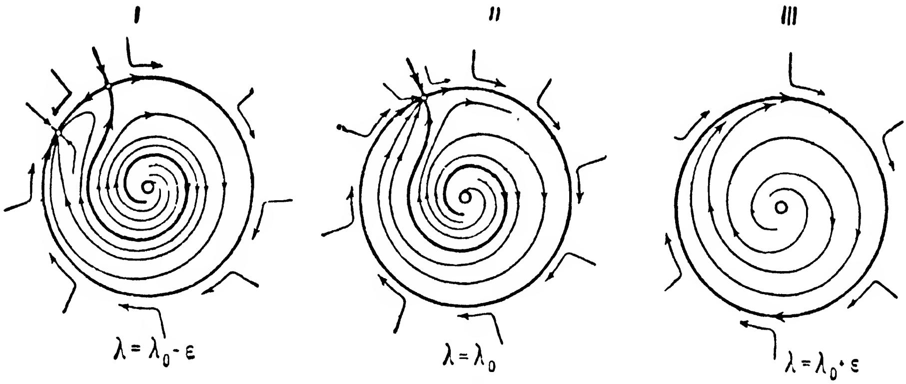


FIG. 201.

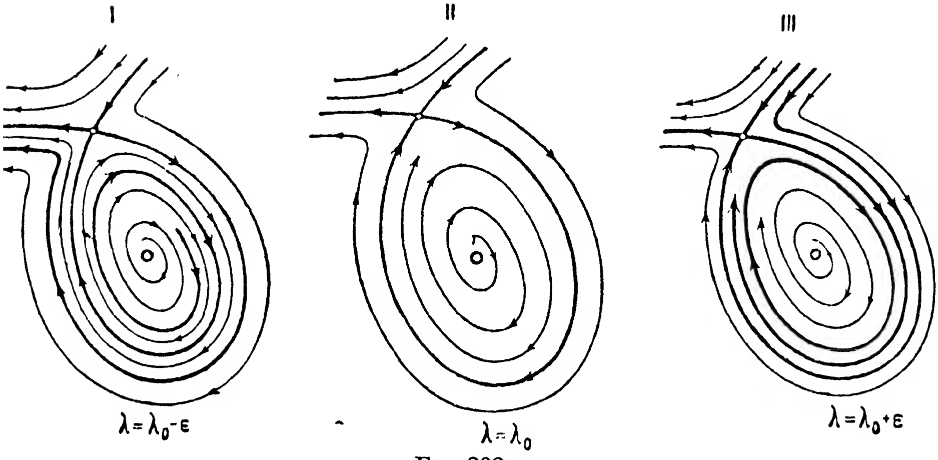


FIG. 202.

(10) will show that they would not have influenced the appearance or the disappearance of the limit-cycle provided that $\bar{\gamma}_1 \neq 0$.

(2) All our conclusions are drawn without taking into account the magnitudes of $R,\bar{\alpha}_1,\bar{\beta}_1,\bar{\gamma}_1$. An analogous but much more complete discussion of the appearance of self-oscillations in a tuned grid oscillator as feedback coupling is increased is found in Chapter IX §2. We will then have to make definite assumptions regarding the magni-

tude of the coefficients of the characteristic of the tube, of the resistance, etc.

2. Appearance of a limit-cycle arising out of a separatrix. This question is of great interest from the point of view of the theory of differential equations and physics in general. Its analysis however presents certain difficulties which have not as yet been overcome. Owing to this we merely give two sketches (Figs. 201 and 202) which illustrate two typical cases. It is easy to see that on each of these drawings part II corresponds to the branch point of the parameter.

§3. Systems without Limit-Cycles

It is often important to know that no limit-cycles can exist. There are available for the purpose a few sufficiency conditions which are quite useful in some cases.

Bendixson's criterion. *If the curve* $\partial P/\partial x + \partial Q/\partial y = 0$ *has no real branches, or which is the same, if* $(\partial P/\partial x + \partial Q/\partial y)$ *has a fixed sign, then there are no limit-cycles, and even no simple closed curve* γ *made up of paths.*

The proof is immediate. Suppose a curve such as γ existed. By the well known theorem of Gauss if Ω is the region bounded by γ then

$$\int\int_{\Omega}\left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}\right) dx\, dy = \oint_{\gamma} (P\, dy - Q\, dx) \neq 0.$$

However along the paths the system (1) holds and so $P\, dy - Q\, dx = 0$. Hence the simple integral is zero. This contradiction proves the criterion.

A few conditions based on the properties of the index hold regarding limit-cycles. Namely limit-cycles are excluded whenever one of the following conditions is fulfilled:

1. There are no singular points.
2. There is only one singular point of index $\neq +1$ (for instance a saddle point).
3. There are several singular points but no collection of them has a sum of indices equal to $+1$.
4. There are only simple singular points but all those forming collections whose sum of indices are $+1$ are limit-points of paths going to infinity.

1. Example: The triggered sweep circuit. We will discuss the so-called triggered sweep circuit used to switch in a displacement on the time axis on the cathode ray oscilloscope. In this oscilloscope a

thin beam of electrons passes between two pairs of deflecting plates. The voltage to be studied is impressed between one pair; a voltage approximately proportional to time is applied between the other pair, and the first voltage is presented on a fluorescent screen as a function of time. If the processes to be observed are very fast, for example in the case of random waves, the "time" voltage must vary rapidly. This rapid change of voltage is obtained by discharging a condenser

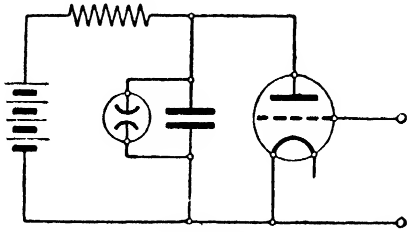


FIG. 203.

through a resistance. In order to initiate this short process at the proper time, i.e. to start the discharge at the time of arrival of the wave, one proceeds in the following way. The pair of plates, whose voltage serves to deflect the electron beam with respect to time, is connected to the plate circuit of a tube. The grid of the tube is maintained at such a high negative voltage that the tube is non-conducting and the condenser across the oscilloscope plates cannot be discharged. This negative voltage blocking the tube can be changed by means of

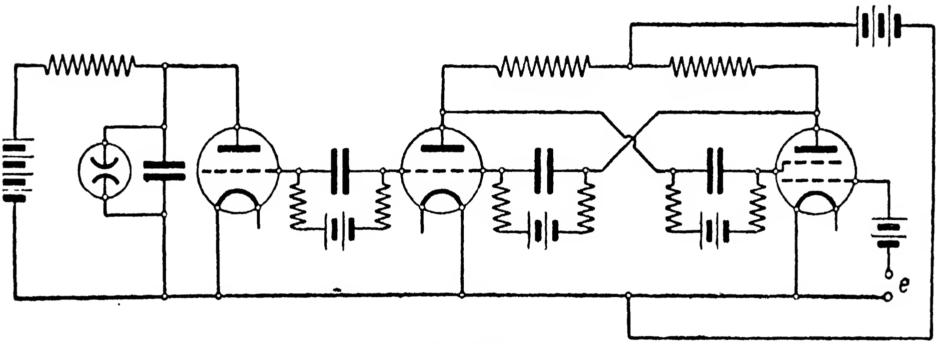


FIG. 204.

a vacuum-tube trigger circuit connected to the grid circuit and "working" under the effect of the impulse of the incoming wave. Since a certain time is required to start the trigger circuit, it is necessary to delay the wave. This is achieved in the following manner: between the relay and the oscilloscope the wave must pass through a transmission line whose length is so chosen that the time delay along the line

enables the trigger circuit to start functioning before the wave arrives at the oscilloscope and thus to eliminate the negative voltage blocking the tube. There exist two types of triggered sweep circuits: one regulates the negative, the other the positive voltages. We shall examine only the first type. The complete circuit is represented in Figure 204. In order to study its theory we shall use a simplified scheme (Fig. 205). The capacitances C represent plate-to-cathode interelectrode capacitances of the tube. For the sake of simplicity we shall assume that the circuit is symmetrical, i.e. that both tubes

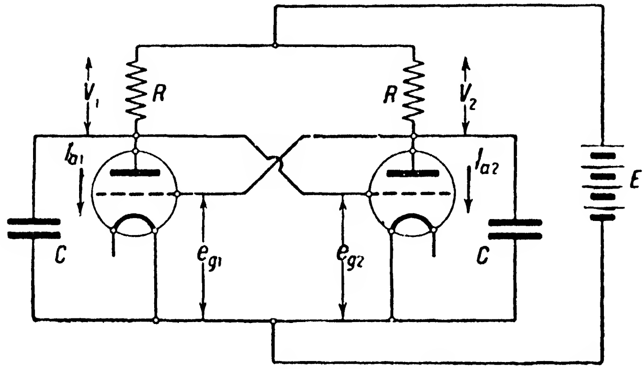


FIG. 205.

and both resistances are identical. The equations corresponding to this are:

$$\dot{V}_1 + \frac{\dfrac{V_1}{R} - I_{a1}}{C} = 0; \qquad \dot{V}_2 + \frac{\dfrac{V_2}{R} - I_{a2}}{C} = 0 \tag{13}$$

where I_{a1} and I_{a2} are the plate currents corresponding to the tubes I and II, V_1 and V_2, the voltage drop in the plate resistances calculated with respect to "zero voltage drop," $V_0 = Ri_0$ (i_0 = plate current). We shall neglect the effect of changes in plate voltage and shall assume that the plate current is a function of the grid voltage alone (the introduction of the plate reaction does not bring anything fundamentally new into our investigation). The transfer characteristics of the tubes are then of the form $I_{a1} = f(e_{g1})$, $I_{a2} = f(e_{g2})$ where e_{g1} and e_{g2} are the grid voltages of the tubes I and II. We shall assume in particular that the characteristic $f(e_g)$ has the form represented in Fig. 206, i.e. that it possesses the following properties:

1. $f(-e_g) = -f(e_g)$, and consequently $f(0) = 0$;
2. $(\partial f/\partial e_g)_0 = g_m > 0$;

3. $\partial f/\partial e_g > 0$, where if $|a| > |b|$ then $(\partial f/\partial e_g)_a < (\partial f/\partial e_g)_b$;

4. And, finally, $f(e_g) < J_g$, i.e. the characteristic is limited and there exists a saturation current.

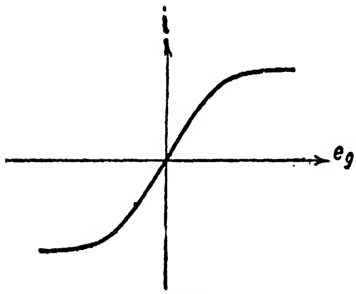


FIG. 206.

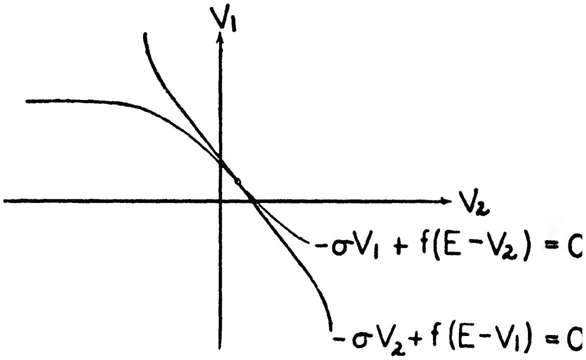


FIG. 207.

Using the characteristic, the initial system (13) reduces to

$$(14)\quad C\dot{V}_1 = -\sigma V_1 + f(E - V_2); \qquad C\dot{V}_2 = -\sigma V_2 + f(E - V_1)$$

where $\sigma = 1/R$ = plate conductance. The equilibrium states are defined by the equations:

$$(15)\qquad -\sigma V_1 + f(E - V_2) = 0$$
$$(16)\qquad -\sigma V_2 + f(E - V_1) = 0$$

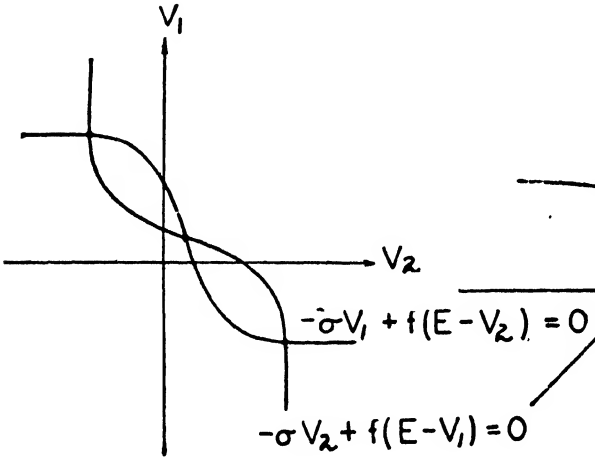


FIG. 208.

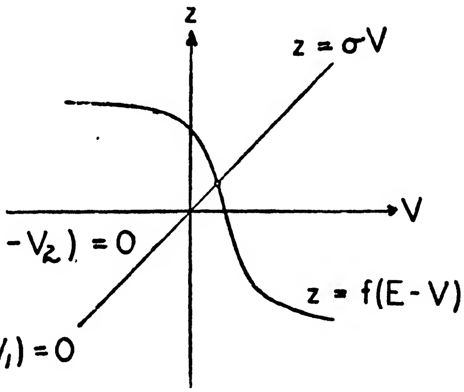


FIG. 209.

The solution of these equations is represented by the intersection points of the curves (15) and (16). There can exist either one (Fig. 207) or three intersection points (Fig. 208). One of the intersection

points, which is always present, is determined by the condition $V_1 = V_2$. In fact, equation:

$$-\sigma V + f(E - V) = 0 \tag{17}$$

has always one and only one solution V_0 defined by the intersection of the straight line $y = \sigma V$ with the curve $y = f(E - V)$ (Fig. 209), and so (V_0, V_0) satisfies (15) and (16).

The three points of intersection of (15) and (16) will coincide with (V_0, V_0) whenever $f'(E - V_0) = \sigma$ and there will be only one intersection namely (V_0, V_0), whenever $f'(E - V_0) < \sigma$.

Let $A(V_1^*, V_2^*)$ be any one of the three intersections. To determine its stability we pass to the related first approximation. Setting $V_1 = V_1^* + V_1'$, $V_2 = V_2^* + V_2'$, the first approximation is found to be

$$C\dot{V}_1' = -\sigma V_1' - f'(E - V_2^*)V_2', \qquad C\dot{V}_2' = -f'(E - V_1^*)V_1' - \sigma V_2'.$$

The related characteristic equation is

$$\lambda^2 + 2\sigma\lambda + \sigma^2 - f'(E - V_1^*)f'(E - V_2^*) = 0,$$

whose roots are

$$\lambda_{1,2} = -\sigma \pm \sqrt{f'(E - V_1^*)f'(E - V_2^*)}$$

and therefore always real. The special point (V_0, V_0) will be a stable node when $f'(E - V_0) < \sigma$ and a saddle point and hence unstable when the opposite inequality takes place. Hence, this equilibrium state is stable when it is unique and is unstable if there exist three equilibrium states. The other two equilibrium states are in this case always stable; they represent stable nodes. It is easy to see by Bendixson's criterion that there are no closed curves consisting of whole paths, for

$$C\left(\frac{\partial P}{\partial V_1} + \frac{\partial Q}{\partial V_2}\right) = -2\sigma \neq 0.$$

On the other hand, the straight line $V_1 = V_2$ is a path. It is also easy to see that all the curves coming from infinity are directed inwards (see p. 238). In fact, according to (14) we have:

$$C\frac{d}{dt}(V_1^2 + V_2^2) = -2\sigma(V_1^2 + V_2^2) + 2V_1 f(E - V_2) + 2V_2 f(E - V_1)$$

so that when V_1 and V_2 are sufficiently large, the right-hand side of the equation is always negative. The cases:

$$f'(E - V_0) < \sigma \qquad \text{and} \qquad f'(E - V_0) > \sigma$$

are represented in Figs. 210 and 211. Equation (14) indicates directly that the curve (15) is the isocline $dV_1/dV_2 = 0$ and the curve (16), the isocline $dV_1/dV_2 = \infty$.

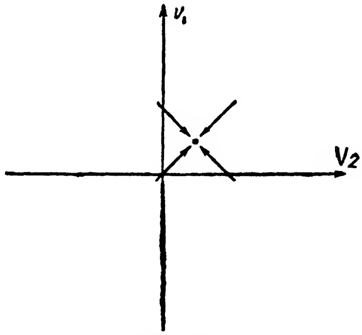


FIG. 210.

Let us investigate what will happen if we suddenly switch in the negative voltage $-e$, which we shall consider as being constant. This additional voltage is applied to the grid of one of the tubes or to another special grid (Fig. 204) or to the same grid to which e_g is

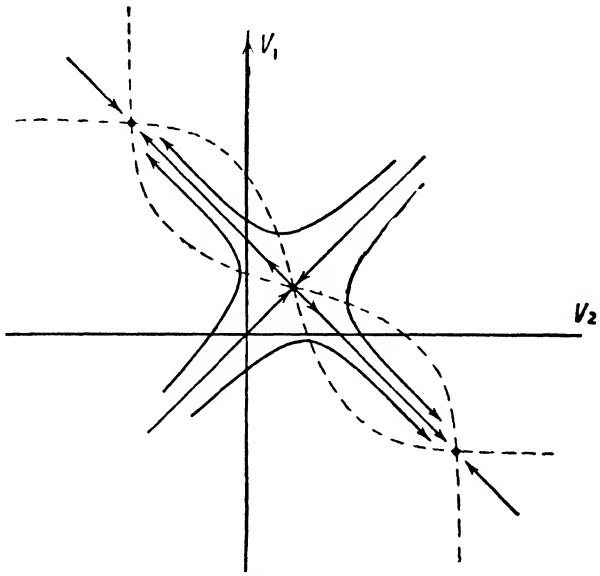


FIG. 211.

applied. In place of (14) we now have

$$C\dot{V}_1 = -\sigma V_1 + f(E - V_2); \qquad C\dot{V}_2 = -\sigma V_2 + f(E - V_1 - ke)$$

where k is a coefficient depending on the place of application of the voltage e. If the voltage e is applied to the control grid, as in Fig. 204,

then $k = 1$. The equilibrium position is determined by the intersection points of the curves:

$$-\sigma V_1 + f(E - V_2) = 0,$$
$$-\sigma V_2 + f(E - e - V_1) = 0.$$

We can see again that there may exist either one or three intersection points. It is easy to obtain the same results as before regarding stability: if there is only one equilibrium position (one intersection point), it is a stable node, and if there are three, then one of them is a saddle point and the other two are stable nodes (Fig. 212). The operation of the triggered sweep circuit may now be described as follows. Let us assume that the signal, to which the relay must

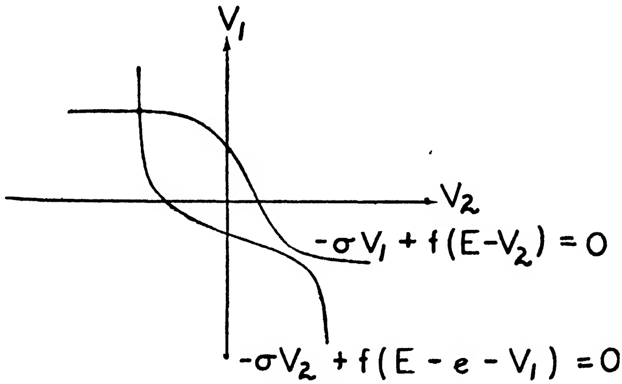


FIG. 212.

respond, has a rectangular form. When the relay starts operating, it is in the equilibrium state corresponding to the negative value $E - V_2$, i.e. to the negative voltage on the grid of the tube I; (the left node in Fig. 211). Therefore, the time deflecting condenser cannot be discharged. The parameters of the system are so chosen as to have the case represented in Fig. 210 at the beginning of the signal. Since there exists only one stable equilibrium state corresponding to the positive voltage on the grid of the tube I, the tube conducts when the signal is applied and the condenser is discharged. At the end of the signal there are again three equilibrium positions but the relay is now in the equilibrium position corresponding to the positive voltage on the grid of the tube I. Thus the signal opens the tube through which the condenser is being discharged, and when the signal stops, the tube remains open.

2. Connection of d.c. generators. Consider two identical d.c. generators both excited in series and which are to be connected in

parallel (Fig. 213). Their common characteristic $E = \psi(i)$ is shown in the first quadrant (Fig. 214) and one must bear in mind that $\psi(-i) = -\psi(i)$. We will set $\rho = \psi'(0)$. Owing to symmetry $r_1 = r_2 = r =$ the common interval resistance of the generators and

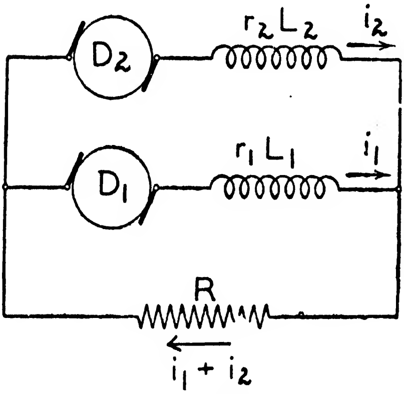


Fig. 213.

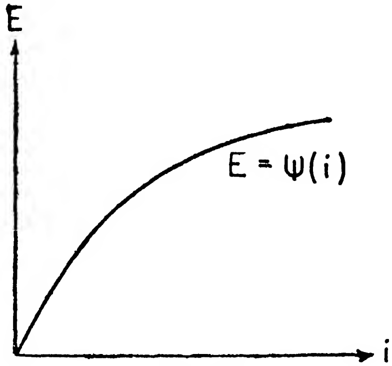


Fig. 214.

$L_1 = L_2 = L =$ their common field inductance. The basic equations are

$$L\dot{i}_1 = -Ri_2 - (R + r)i_1 + \psi(i_1)$$
$$L\dot{i}_2 = -Ri_1 - (R + r)i_2 + \psi(i_2)$$

The equilibrium states are given by

$$Ri_1 + (R + r)i_2 - \psi(i_2) = 0, \qquad Ri_2 + (R + r)i_1 - \psi(i_1) = 0.$$

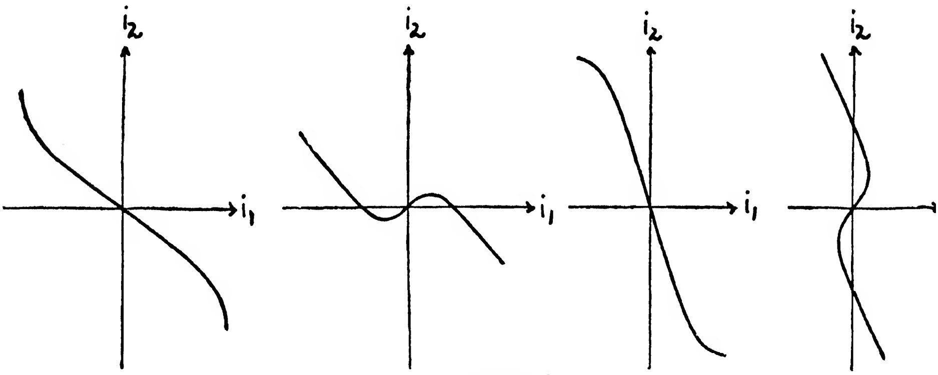


Fig. 215.

Clearly the origin $i_1 = i_2 = 0$ is among them. The two curves whose intersection gives rise to the singular points are symmetrical to one another with respect to the bisector $i_1 = i_2$ of the first quadrant. Various possibilities for one of them are indicated in Fig. 215, and

various intersections, i.e. possibilities for the singular points, are shown in Fig. 216. The most complicated case is the one where there are nine singular points. We shall determine their nature.

Origin. The equations of first approximation are

$$\begin{aligned} L\dot{i}_1 &= (\rho - R - r)i_1 - Ri_2 \\ L\dot{i}_2 &= -Ri_1 + (\rho - R - r)i_2. \end{aligned}$$

The characteristic equation is

$$(\lambda + r + R - \rho)^2 - R^2 = 0$$

and hence the characteristic roots are $\rho - r$ and $\rho - r - 2R$. The

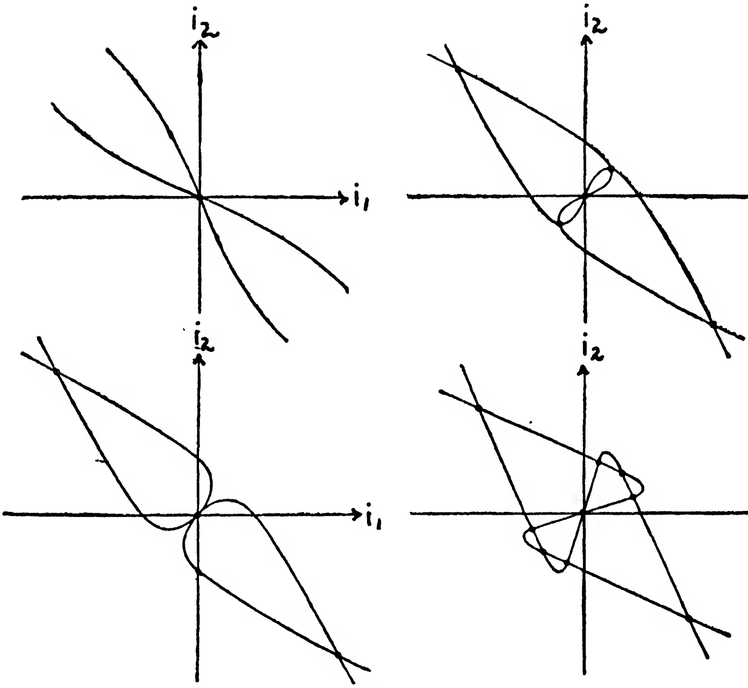


FIG. 216.

origin is stable when $r > \rho$ and unstable when $r < \rho$. It is a saddle point when $\rho - 2R < r < \rho$ and an unstable node when $r < \rho - 2R$.

Points other than the origin on $i_1 = -i_2$. These are the harmful points since in these points $I = i_1 + i_2 = 0$: no current goes to the line; one of the machines is a generator and the other a motor. Let $i_1 = -i_2 = i_0$, and set $\rho_0 = \psi'(i_0)$. The equations of first approximation are as above with ρ replaced by ρ_0. Since i_0 is rather large, ρ_0 is small and so this time $\rho_0 < r$, both λ_1 and λ_2 are negative and we have a stable node. Thus the two harmful states are both stable.

Points other than the origin on $i_1 = i_2$. They are defined by $\psi(i) = (r + 2R)i$, i.e. by the intersection of the curve $y = \psi(i)$ with the line $y = (r + 2R)i$. They cannot occur unless the slope of the line is smaller than the slope of the characteristic at the origin, i.e. unless $r + 2R < \rho$. Let $i_1 = i_2 = i_0'$ in the present case and set $\rho_0' = \psi'(i_0')$. The same calculation of the characteristic roots yields here the values $\rho_0' - r$, $\rho_0' - r - 2R$. Thus both are real and foci are excluded. If i_0' is quite small ρ_0' will be very near ρ, both roots will be positive and we will have an unstable node. As we shall

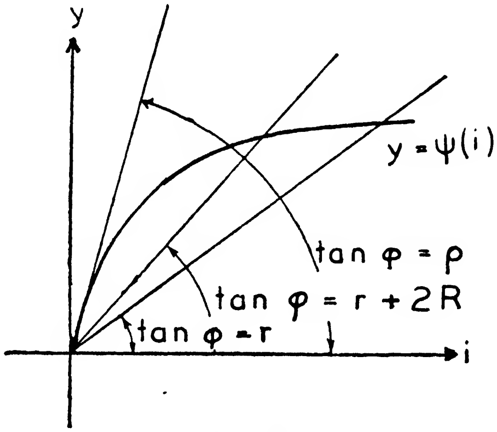


FIG. 217.

see the points in question can be shown to be nodes without any calculation.

Points not on the bisectors $i_1 = \pm i_2$. It will be shown (§4) that all the motions "run away" from infinity, and that on a very large circumference C the vectors $(\dot{i}_1, \dot{i}_2)$ all point inwards or all outwards. Hence the Poincaré index of C is $+1$ (Chap. V, §8). Since we have only simple singular points the four unknown points have the same index $\epsilon = \pm 1$. Let also $\eta = \pm 1$ be the common index of the two points other than the origin on the bisector $i_1 = i_2$. The origin and the two unfavorable points on the line $i_1 = -i_2$ are nodes and hence their common index is $+1$. Writing down that the index of C is the sum of those of the singular points which it surrounds we have the relation

$$3 + 2\eta + 4\epsilon = 1$$

The only values of η and ϵ compatible with $\eta = \pm 1$, $\epsilon = \pm 1$ are $\eta = 1$, $\epsilon = -1$. Hence the points on $i_1 = i_2$ are nodes and the four points not on the bisectors are saddle points.

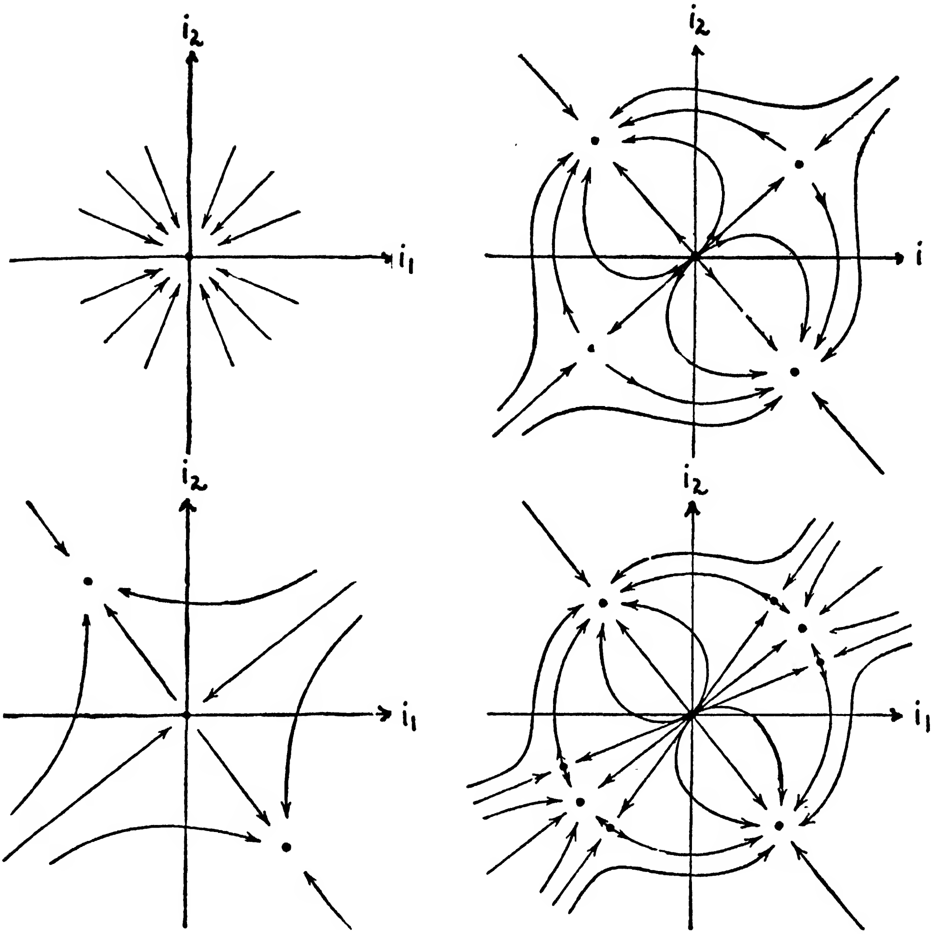


FIG. 218.

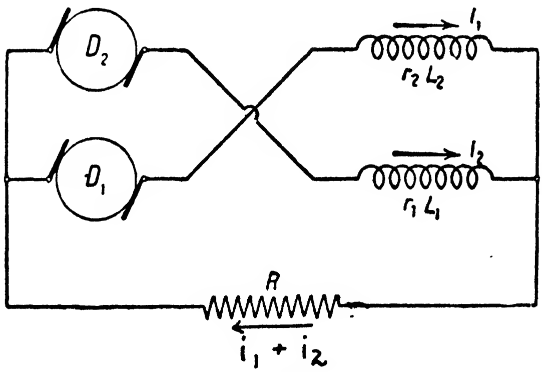


FIG. 219.

The various admissible phase portraits are sketched in Fig. 218.

The following conclusion, which did not require the full analysis of the singular points, may be stated: In view of the stability of the harmful state one should not connect in parallel generators excited in series. The connection of Fig. 219 is however correct with the field coils crosswise. The equilibrium state with the current flowing outside is then found to be stable and the harmful state unstable.

§4. BEHAVIOR OF THE PATHS AT INFINITY

By "behavior at infinity" we mean far out in the phase plane. This behavior is often quite helpful in obtaining the full phase portrait. Sometimes also it is found quite readily. Thus from (1) there follows immediately

$$\tfrac{1}{2}dr^2/dt = xP(x,y) + yQ(x,y) = R(x,y).$$

Suppose that outside and on a circle T of large radius the function R has a constant sign. If the sign is $+$ the motions tend to infinity when $t \to +\infty$, and infinity is "stable." If the sign is $-$ the same behavior holds for $t \to -\infty$, and infinity is "unstable."

As an application consider the problem of the two generators just discussed in §3 and let us suppose $\psi(i)$ to be approximated, as we may generally do, by an odd polynomial of degree >1:

$$\psi(i) = i(a_0 + a_2 i^2 + \cdots + a_{2n} i^{2n})$$

The analogue of the expression $xP + yQ$ for the system (1) is here

$$f(i_1,i_2) = i_1\psi(i_1) + i_2\psi(i_2) - 2Ri_1i_2 - (R + r)(i_1^2 + i_2^2)$$

and it is a polynomial in i_1,i_2. Its highest degree terms are

$$g(i_1,i_2) = a_{2n}(i_1^{2n+2} + i_2^{2n+2})$$

and the sign of f far from the origin is the sign of g, i.e. the sign of a_{2n}, and consequently fixed. Suppose $a_{2n} < 0$. Then along a circle C of large radius ρ, $\dot{r}^2$ will be negative, r^2 will decrease and so will r. That is to say if M is any point of C, the motion starting from M must be such that r decreases. Hence it will be inwards. Thus the velocity vector (P,Q) will point inwards at all points of C. When $a_{2n} > 0$ the situation is the same save that the vector always points outwards along C. In the first case infinity is unstable, in the second it is stable. This is the proof of the statement regarding infinity made in the treatment of the problem of the two generators in §3.

Returning to the general problem of the behavior at infinity, it

cannot generally be dealt with in the simple manner just described and one has to apply a suitable change of variables. The most satisfactory procedure is one used by Poincaré. Geometrically it consists in a central projection of the x,y plane on a sphere of radius one tangent to the plane at the origin (Fig. 220). To a point N on the plane there correspond two points N_1 and N_2 on the sphere, situated on the line O_1N passing through the center of the sphere. Thus the plane is represented twice: on the lower (southern) and on the upper (northern) hemisphere. The infinitely remote points of the plane are projected on the equator (large circle parallel to the x,y plane). The straight

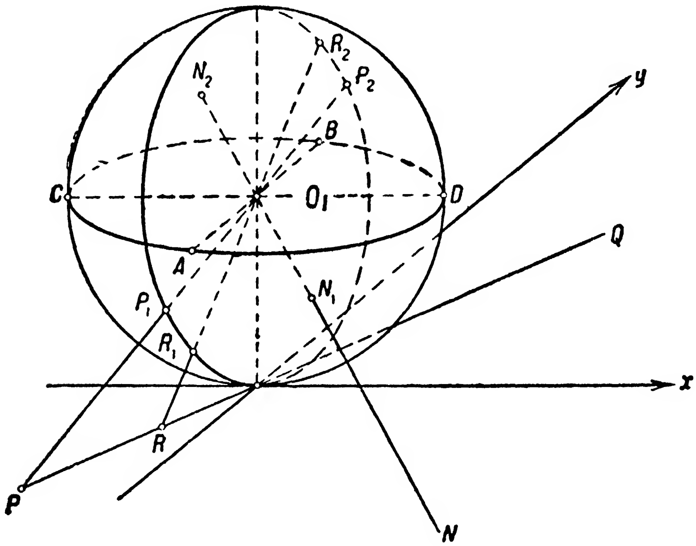


FIG. 220.

lines of the plane will be transformed into great circles of the sphere; the straight lines passing through the origin will become large circles perpendicular to the equator. For example the line PR becomes a large circle passing through the points P_1R_1,P_2R_2. The paths in the plane will be transformed into corresponding curves on the sphere; the saddle points, the nodes, and the foci will preserve their nature. New singular points may however arise on the equator. For example the images of the paths for which $y \to 0$ far from the origin pass through the points C,D and those for which $x \to 0$ pass correspondingly through the points A,B. Thus *the singular points on the equator need not correspond to the intersections of the curves* $P(x,y) = 0$, $Q(x,y) = 0$. They characterize the behavior of the paths infinitely far from the origin.

We must now obtain analytical expressions for the transformation. To that end let u,v,z denote cartesian coordinates for the sphere with the u and v axes parallel to the x and y axes and with the z axis vertical. Thus we will have

$$u^2 + v^2 + z^2 = 1,$$

and at the same time since $N(x,y,1)$ and $N_1(u,v,z)$ are collinear

$$\frac{u}{x} = \frac{v}{y} = \frac{z}{1} = \frac{1}{\epsilon\sqrt{x^2 + y^2 + 1}}, \qquad \epsilon = \pm 1.$$

The two signs of ϵ correspond to the two points N_1,N_2. As a consequence

$$u = \frac{x}{\epsilon\sqrt{x^2 + y^2 + 1}}, \; v = \frac{y}{\epsilon\sqrt{x^2 + y^2 + 1}}, \; z = \frac{1}{\epsilon\sqrt{x^2 + y^2 + 1}}.$$

In order to have again a relation from plane to plane it is convenient to project on one of the planes $u = 1$ or $v = 1$, analogous to the initial plane $z = 1$. Let us project for instance on $v = 1$. The coordinates will then be u,z for that plane and the corresponding relations are:

$$u = \frac{x}{y}, \; z = \frac{1}{y},$$

and hence also

(18) $$x = \frac{u}{z}, \qquad y = \frac{1}{z}.$$

This transformation from the plane x,y to the plane u,z is particularly suitable for discussing the nature of all the points except A and B (Fig. 220). For the points A,B the more advantageous transformation is to $u = 1$, with v,z as the coordinates and relations

(19) $$x = \frac{1}{z}, \qquad y = \frac{v}{z}.$$

It is again clear that both transformations (18), (19) preserve the character of the singular points.

Let us apply transformation (19) to the fundamental system (1). We first find

$$dx = -\frac{dz}{z^2}, \qquad dy = \frac{z\,dv - v\,dz}{z^2}$$

and hence

$$\text{(20)} \qquad \dot{z} = -P\left(\frac{1}{z}, \frac{v}{z}\right) z^2, \qquad \dot{v} = -P\left(\frac{1}{z}, \frac{v}{z}\right) vz + Q\left(\frac{1}{z}, \frac{v}{z}\right) z.$$

Let p,q be the degree of $P(x,y),Q(x,y)$. Then we may write

$$P\left(\frac{1}{z}, \frac{v}{z}\right) = z^{-p}P^*(z,v), \qquad Q\left(\frac{1}{z}, \frac{v}{z}\right) = z^{-q}Q^*(z,v)$$

where P^*,Q^* are polynomials: Suppose $q \geqslant p$, and let us introduce a new time τ by the relation

$$d\tau = x^{q-1}\, dt = z^{-q+1}\, dt.$$

Thus τ increases with t when $x > 0$ and may decrease when t increases whenever $x < 0$. Hence to the right of the y axis the "τ motions" along the paths will proceed in the same direction as the "t motions" (i.e. as the true motions) and to the left of the y axis the τ motions may proceed in the opposite direction from the true motions.

We now have from (20) the new system

$$\text{(21)} \qquad \begin{cases} \dfrac{dz}{d\tau} = -z^{q-p+1}P^*(z,v), \\ \dfrac{dv}{d\tau} = -vz^{q-p}P^*(z,v) + Q^*(z,v). \end{cases}$$

We may well expect new singular points for which $z = 0$, i.e. at infinity. The system (21) is adequate for all those for which z and v are both finite. Since $v = y/x$, v finite means "not in the direction of the y axis." In other words (21) is adequate for all the singular points on the equator other than A and B.

Similarly to (18) there corresponds a system which is suitable for all equatorial points except C and D:

$$\text{(22)} \qquad \begin{cases} \dfrac{dz}{d\tau} = -zQ^{**}(z,u) \\ \dfrac{du}{d\tau} = z^{q-p}P^{**}(z,u) - uQ^{**}(z,u) \end{cases}$$

Application to a linear oscillator with friction. The basic differential equation is

$$\ddot{x} + 2h\dot{x} + \omega^2 x = 0$$

and it reduces to the system

$$\dot{x} = y = P(x,y), \qquad \dot{y} = -\omega^2 x - 2hy = Q(x,y).$$

The related systems (21), (22) are found to be

$$(21') \qquad \frac{dz}{d\tau} = -zv, \qquad \frac{dv}{d\tau} = -\omega^2 - 2hv - v^2;$$

$$(22') \qquad \frac{dz}{d\tau} = 2hz + \omega^2 uz, \qquad \frac{du}{d\tau} = 1 + 2hu + \omega^2 u^2.$$

There are no singular points on $u = 0$, $v = 0$, while those of the first system are given by

$$z = 0, \qquad v^2 + 2hv + \omega^2 = 0.$$

From the first of (21′) or (22′) we see that in any case the equator $z = 0$ is a path or else composed of separatrices (from singular point to singular point). There are two possibilities:

I. $h^2 < \omega^2$. Then there are no singular points on the equator and the latter is a limit-cycle.

II. $h^2 > \omega^2$. There are then four singular points on the equator at the intersections with the planes $v = v_1$, $v = v_2$ where

$$v_1 = -h + \sqrt{h^2 - \omega^2}, \qquad v_2 = -h - \sqrt{h^2 - \omega^2}.$$

Let $z = 0$, $v = v_i$ be one of the points. To determine its stability set $z = 0 + \xi$, $v = v_i + \eta$. Inserting in (21′) there results the first approximation system

$$\dot{\xi} = -v_i \xi, \qquad \dot{\eta} = -2(v_i + h)\eta$$

The characteristic roots are thus

$$\lambda_1 = -v_i, \qquad \lambda_2 = -2(v_i + h).$$

Thus for v_1 the root $\lambda_1 > 0$ and $\lambda_2 < 0$ hence v_1 is a saddle point. On the other hand for v_2 we find $\lambda_1 > 0$ and $\lambda_2 > 0$ and so v_2 is a node. Its "stability" behavior is readily determined from the behavior of the paths. Since we have here a damped vibration the origin is always stable, the motions all tend away from infinity and so infinity is unstable.

Remark. We have supposed $q \geqslant p$. When $p < q$ the required modifications are obvious.

Figure 221 describes the behavior of the paths at infinity for $h^2 < \omega^2$. Figure 222 represents the orthogonal projection of the sphere on the plane tangent to the sphere in the lower point when $h^2 > \omega^2$.

Application to limit-cycles. Suppose that infinity (the equator) is an unstable limit-cycle. If there is just one singular point at finite distance and it is an unstable node or a focus, then on the basis of the general theory described in Chap. V, §7 we must necessarily have at

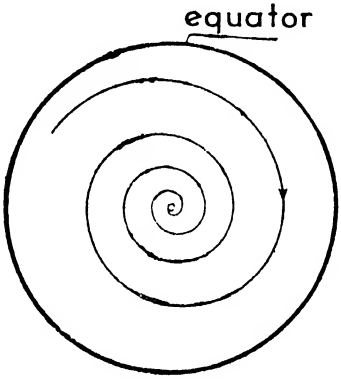


FIG. 221.

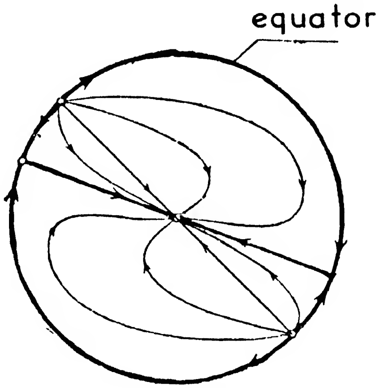


FIG. 222.

least one stable limit-cycle. This concept will be used to prove the existence of a limit-cycle in the simplest self-oscillating scheme with a tuned grid (see Chap. IX, §2 for more details). Kirchhoff's equations yield here

$$L\dot{i} + Ri + \int \frac{i\,dt}{C} = M\dot{I}_a.$$

We will assume that the characteristic of the tube is adequately expressed by

$$I_a = I_0 + \frac{2I_s}{\pi} \arctan \frac{g_0\pi}{2I_s} v_g,$$

where I_s is the saturation current, g_0 the mutual conductance of the tube at the operating point and $v_g = \int i\,dt/C$ the grid voltage. It is not difficult to show that the general theory applies here also. The equation of motion written in the ordinary way will take the form:

$$(23)\quad \dot{v}_g = \frac{i}{C} = P(v_g,i); \qquad \dot{i} = -\frac{R}{L}i - \frac{v_g}{L} + \frac{Mg_0 i}{LC\left(1 + \frac{g_0^2\pi^2 v_g^2}{4I_s^2}\right)} = Q(v_g,i).$$

It is easy to see that the character of the equilibrium states situated on the axis $z = 0$ is the same for the system (23) and for the system:

$$\dot{z} = -\frac{zv}{C}; \qquad \dot{v} = -\frac{Rv}{L} - \frac{1}{L} - \frac{v^2}{C}. \tag{24}$$

Since (24) is independent of the characteristic, the behavior of the paths of the initial system at infinity must be the same as that corresponding to a harmonic oscillator whose frequency is $1/\sqrt{LC}$ and whose damping coefficient is R/L. We have investigated this case and we have found that infinity is unstable. Consequently, if the unique singular point, situated at a finite distance is unstable, the

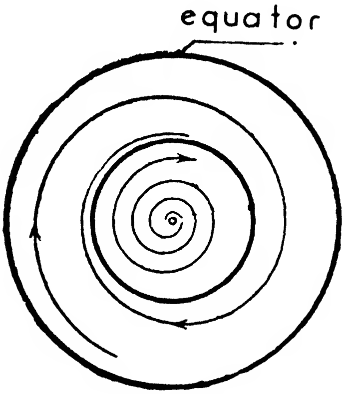


Fig. 223.

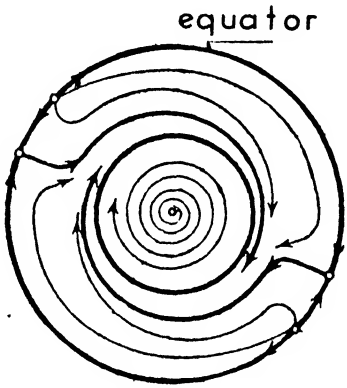


Fig. 224.

system here considered must possess at least one stable limit-cycle. The two possible cases: no singular points on the equator and singular points on the equator together with the corresponding dispositions of the paths are respectively described in Figs. 223 and 224.

To sum up, the consideration of the behavior of the paths at infinity has enabled us to prove the existence of limit-cycles in certain cases. To the physicist such an existence proof may appear strange. He knows experimentally that the vacuum tube oscillator, for instance, can assume steady oscillations, and he needs no special proof of this fact. What has actually been accomplished however is to show that the theory, with the assumed idealization gives results in agreement at an important point with observation. In other words what has actually been obtained is a certain justification of the theory, or if preferred, of the mathematical model.

§5. Position of the Limit-Cycles

This is certainly the most obscure part of the whole theory—indeed no general method for locating limit-cycles is known and all that we have is a few negative results.

Sometimes the following procedure may succeed. Take a group of singular points whose sum of indices is $+1$. A limit-cycle, it will be recalled, can only surround a group of singular points which has this property. Draw around the selected group two closed curves forming a ring-like region Ω free from singular points. Then, if the velocity vector of the representative point in the phase plane of these curves is never directed outside of Ω, we can say that there exists at least one stable limit-cycle within the region. If the velocity vector, is never directed toward the inside, there exists at least one unstable limit-cycle within the region. If we find n regions such as Ω we can affirm that there exist at least n limit-cycles. Similarly, if infinity is unstable and there exists a closed curve along which the velocity vector is always directed outward, and outside of which singular points do not exist, we have at least one stable limit-cycle outside the curve. Similar considerations hold for unstable limit-cycles save that one must consider the opposite direction of the velocity vector.

In some cases a certain method due to Poincaré may be applied. Let us suppose that we have a system of nested ovals

$$F(x,y) = C$$

filling up the plane. To simplify matters assume F analytic throughout the plane. An example is a system of concentric circles or concentric ellipses. Let Γ be the curve of contact or locus of all the points where the velocity vector is tangent to one of the ovals. Let C_1,C_2 be the smallest and largest values of C such that the corresponding ovals meet Γ. Let Ω be the ring bounded by the two curves. It is an elementary matter to show that every limit-cycle γ is tangent to some oval and hence meets Γ. Consequently γ must be in Ω. Thus if any limit-cycles exist they must be in Ω. A sufficient condition for the existence of at least one limit-cycle is that the vector of phase velocity on both curves C_1 and C_2 points either everywhere outside or everywhere inside of Ω.

For our basic system (1) the equation of the curve of contact is

$$\frac{P}{Q} = -\frac{\partial F/\partial y}{\partial F/\partial x}.$$

In the special case when the system of ovals consists of a family of concentric circles with center at the origin, i.e. $F(x,y) = x^2 + y^2 = C$, the equation of the curve of contact becomes:

$$P/Q = -y/x.$$

If the dynamical system is described by equations in polar coordinates, we have:

$$\dot{r} = R(r,\phi), \qquad \phi = \Phi(r,\phi).$$

If the ovals are defined by the equation:

$$\frac{dr}{d\phi} = F(r,\phi),$$

then the equation of the curve Γ is

$$\frac{R(r,\phi)}{\Phi(r,\phi)} = F(r,\phi).$$

When the system of ovals consists of concentric cycles centered at the origin, the equation of Γ is

$$R(r,\phi) = 0.$$

Consequently, in this case, the extreme radii of the circles tangent to Γ are obtained from the relations:

$$\frac{dr}{d\phi} = 0, \qquad R(r,\phi) = 0$$

Consider now any branch Γ' of the curve of contact Γ, and let M be an arbitrary point of Γ'. It may happen that the path γ through M, while tangent to Γ' does actually cross it, much as an inflectional tangent crosses a curve. We say then that γ and Γ' have a contact of *even* order at the point M (an inflexion is a contact of order two) and Γ' must be rejected. For it is clear that the geometric argument breaks down for contacts of even order. Thus one must only preserve the branches corresponding to contacts of odd order (contacts without crossing). We see then that if Γ has no branches of odd contact, and in particular no real branches whatever, there are no limit-cycles in the system.

Application to a vacuum tube oscillator in soft regime. These oscillations will be fully discussed in Chap. IX. It will suffice to say that if the vacuum tube has a cubic characteristic the basic equation is

reducible to the van der Pol type

$$\ddot{x} + \epsilon(x^2 - 1)\dot{x} + x = 0 \tag{25}$$

or equivalently to the system

$$\dot{x} = y, \qquad \dot{y} = -x + \epsilon(1 - x^2)y. \tag{26}$$

Choose as family of ovals the circles

$$x^2 + y^2 = C.$$

The equation of the curve of contact is then

$$y(-x + \epsilon(1 - x^2)y) = -xy$$

or

$$y^2(1 - x^2) = 0.$$

Since $y = 0$ is a double root, one may show that the x axis represents the locus of double contacts (even contacts), and consequently this

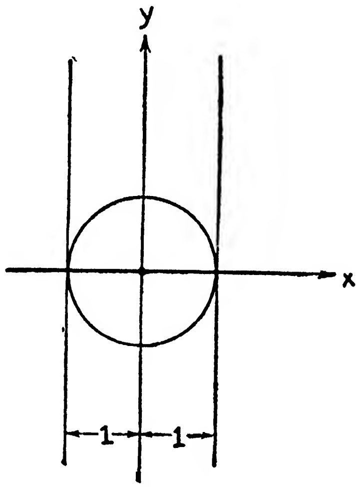


FIG. 225.

branch of the curve may be discarded. Thus we are left with two branches, $x = +1$, and $x = -1$ (Fig. 225). The smallest circle tangent to the curve Γ which is left is a circle of radius 1, and there is no largest circle. Consequently, the limit-cycle, if it exists, is in the region outside the circle of radius 1. Thus the present method enables us to determine, truly not very accurately, the position of the limit-cycle. This result will become obvious later when we examine the physical side of the question, for $x < 1$ corresponds to "negative friction" while $x > 1$ corresponds to positive friction.

Application to a certain pendulum. In this example the phase surface is a cylinder. Namely, we consider the oscillations of a pendulum with friction under the effect of a constant moment. The basic equation is

$$I\ddot{\phi} + mgl \sin \phi + h\dot{\phi} = P$$

where all the constants are positive. The synchronous motor leads to the same type of equation. The phase cylinder corresponds to the coordinates ϕ, $\dot{\phi}$. Choose as the ovals a family of parallels $\dot{\phi}$ = constant on the cylinder. The curve of contact Γ is of the form:

$$\dot{\phi} = \frac{P - mgl \sin \phi}{h}.$$

The upper and lower parallels tangent to Γ are

$$\dot{\phi} = \frac{P + mgl}{h}, \qquad \dot{\phi} = \frac{P - mgl}{h}.$$

Consequently, the limit-cycle, if it exists, is situated within the limits

$$\frac{P - mgl}{h} < \dot{\phi} < \frac{P + mgl}{h}.$$

We know of no physical example of an autonomous system described by two equations of the first order, where the position of the limit-cycles may be at all satisfactorily determined by means of the curve of contact.

It is sometimes possible to prove the presence or the absence of limit-cycles for special types of systems (1) by considerations appropriate to the system. Such an analysis has been given by Liénard for cathode oscillators under certain simplifying assumptions regarding the symmetry of the characteristic.

§6. APPROXIMATE METHODS OF INTEGRATION

Whenever no direct mathematical attack is available one must of necessity have recourse to approximate methods: integrating machines, graphical integration, etc. This is not the place to discuss the machines and we shall only consider rapidly the other methods. More complete information is easily found in the literature; see notably K. Runge, *Graphische Methoden*, B. G. Teubner, 1932. The best graphical procedure is the *method of isoclines*, which consists in the following. The equation of the paths, with time eliminated is

$$\frac{dy}{dx} = \frac{Q(x,y)}{P(x,y)} = f(x,y).$$

The curve $f(x,y) = C$ is the isocline, where the paths have the slope C. By constructing a sufficient number of isoclines (Fig. 226) we can

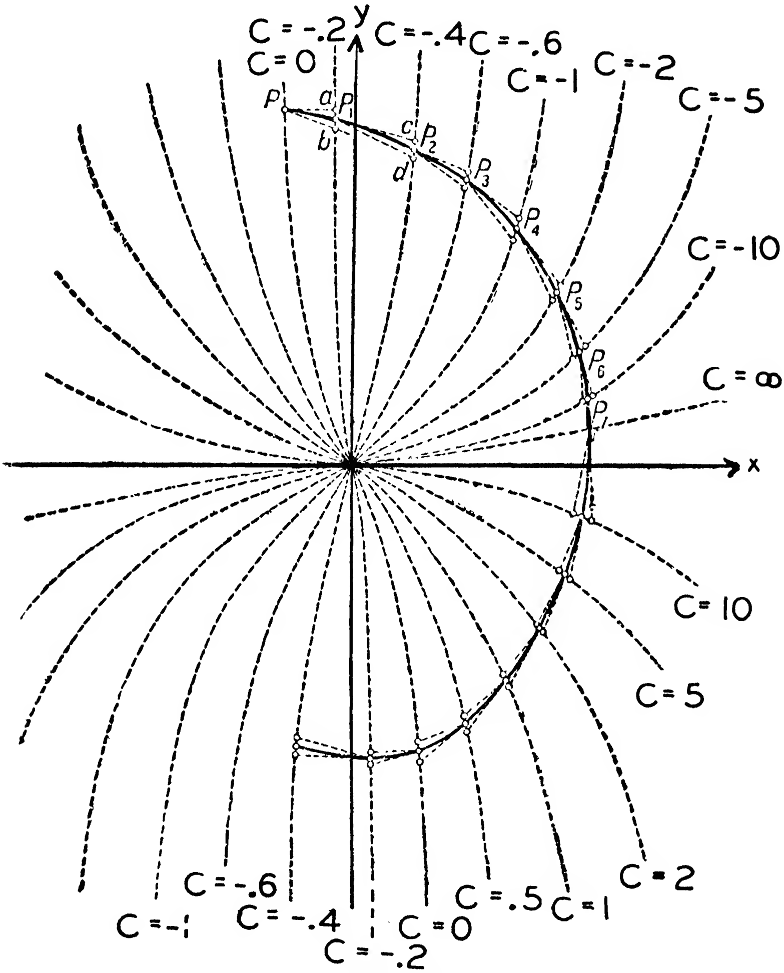


Fig. 226.

begin to construct an approximate phase portrait. Let us start our construction with the path passing through a point P on the isocline $C = 0$. We pass two segments through P, one in the direction of the tangent corresponding to the isocline $C = 0$, and the other in the direction of the tangent corresponding to the adjacent isocline $C = -0.2$. We extend these segments up to their intersection with this adjacent isocline. We thus obtain two points: a and b, and their mid-point P_1

is the second point of our path. From the point P_1 we draw two straight lines up to their intersections c,d with the isocline $C = -0.4$, with inclinations corresponding to the isoclines $C = -0.2$ and $C = -0.4$. Then the mid-point P_2 of cd is the third point of the path for which we are looking. If we continue this construction, we obtain a sequence of points P,P_1,P_2,P_3,P_4 . . . ; and can pass through these points a path which will pass through P. By repetition one may thus obtain a series of paths, and hence an approximate but sufficiently detailed phase portrait of the system under investigation. This

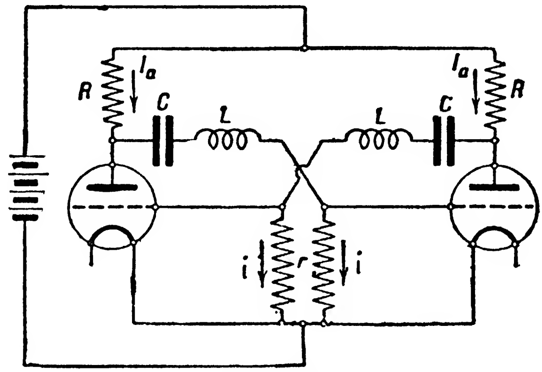


FIG. 227.

assumes of course that its parameters are given definite numerical values. This portrait will allow us to judge whether self-oscillations can arise in the system for the given values of the parameters and to estimate the maximum values reached by x and y during these oscillations, etc. It does not enable us however to determine the variation of the behavior of the system as a function of one of its parameters. In order to answer this question we must construct an entire "gallery" of phase portraits corresponding to different values of the parameter.

A typical example illustrating the application of the isocline method, is found in the investigation of the phase portrait of the van der Pol equation (25) performed by van der Pol himself. This equation arises in many questions related to self-oscillations. For example, the equation of a vacuum tube oscillator possessing a cubic characteristic is reducible to that type. Van der Pol was interested in this equation in connection with the theory of the relaxation oscillations of a symmetric multivibrator with inductance (Fig. 227). See also Chap. VIII. Replacing the van der Pol equation by (26) we have at once

$$\frac{dy}{dx} = +\epsilon(1 - x^2) - \frac{x}{y},$$

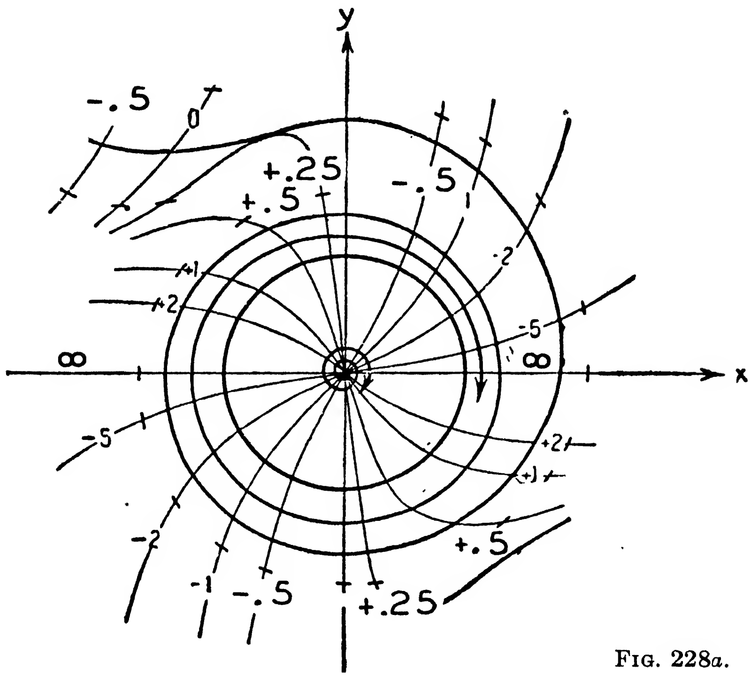


Fig. 228a.

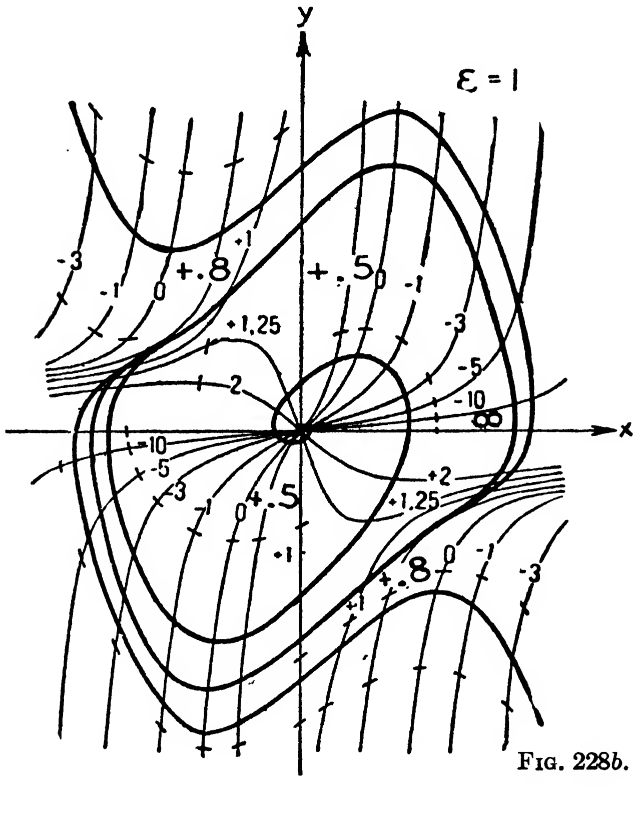


Fig. 228b.

By taking definite numerical values for ϵ and applying the isocline method, van der Pol obtains the "phase portrait gallery" of Fig. 228. Since all the portraits contain a limit-cycle, self-oscillations exist for all values of ϵ. When ϵ increases, the coefficient of the term containing

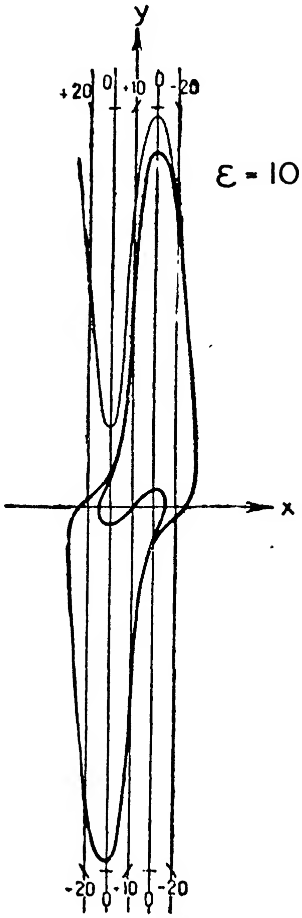


FIG. 228c.

$\dot{x}$ increases, the form of the oscillations differs more and more from the sinusoid (the form of the limit-cycle differs more and more from a circle) and finally the initial growth of oscillations, which is oscillatory when ϵ is small (the singular point is an unstable focus), becomes aperiodic (the singular point is an unstable node) when ϵ is large.

CHAPTER VII

Discontinuous Oscillations. Parasitic Parameters and Stability

In this chapter we shall mainly consider various types of discontinuous oscillations arising in autonomous systems

$$\dot{x} = P(x,y), \qquad \dot{y} = Q(x,y). \tag{1}$$

This time however P,Q will generally not be continuous throughout the whole plane. Needless to say in regions where they are analytic the general results regarding singular points and limit-cycles will be valid and will be applied without hesitation. In the last section we shall discuss certain related questions of degeneracy caused by stray impedance.

§1. Systems with One Degree of Freedom Described by Two Equations of the First Order

We have seen that the investigation of a self-oscillatory system is simplified when one of the oscillatory parameters plays a secondary role and can be neglected. This may lead generally to the lowering of the order of the equation and to the possible existence of discontinuous oscillations. It may well happen however that through disregarding certain parameters discontinuous solutions become admissible but that the order of the equation remains unchanged so that we still have a system with two oscillatory parameters. For example the circuit of Fig. 229, if one disregards all the inductances, goes into the familiar scheme of Fig. 230, described by one differential equation of the first order admitting discontinuous solutions. If only L_1 and L_2 are small (and can be neglected), while L is large and must be taken into account, but the resistance R is small, we obtain the circuit of Fig. 231 studied earlier (Chap. I, §7). The corresponding differential equation is of the second order and not of the first. In spite of the inductance L in Fig. 231 abrupt changes of current in the circuit of the capacitor C and the resistance R are possible, if they are such that the current I through L, and the voltage V across C remain constant. This condition can be fulfilled and consequently discontinuous oscillations may arise. Thus neglecting L, for example, does not diminish the order of the equation but allows the application of

the general considerations relative to discontinuous oscillations, which simplify the treatment. If we neglect the grid currents and the plate

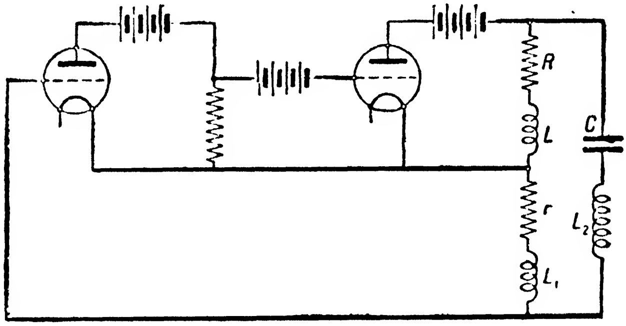


Fig. 229.

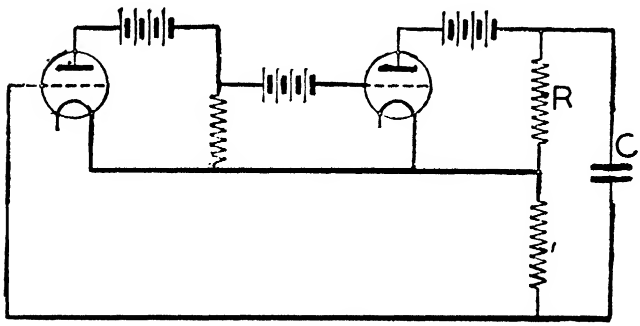


Fig. 230.

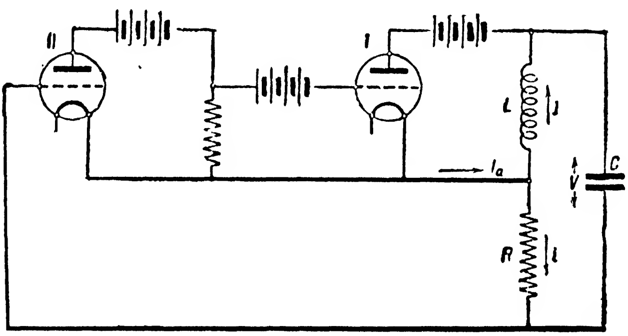


Fig. 231.

reaction, we obtain, for the scheme under investigation (see Chap. I, §7) the following equations:

$$
\text{(2)} \qquad \begin{cases} I = \phi(kri) - i; \\ L\dot{I} - ri - \dfrac{1}{C}\displaystyle\int i\,dt = 0 \end{cases}
$$

where ϕ is the transfer characteristic for the tube I, and k the amplification coefficient for the tube II (we assume again that the tube II operates in the linear part of its characteristic). Let us set

$$kri = x, \qquad \dot{I} = y,$$

so that x is the grid voltage of the tube. Then our equations can be written

$$Ly - \frac{1}{k}x - \frac{1}{krC}\int x\,dt = 0, \qquad \phi(x) - \frac{x}{kr} - \int y\,dt = 0.$$

Differentiating both equations we obtain after a simple transformation

$$\text{(3)} \qquad \begin{cases} \dot{x} = \dfrac{y}{\left(\phi'(x) - \dfrac{1}{kr}\right)} \\ \dot{y} = \dfrac{x}{krLC} + \dfrac{1}{kL}\dfrac{y}{\left(\phi'(x) - \dfrac{1}{kr}\right)}. \end{cases}$$

To simplify matters we make the following very natural assumptions regarding the characteristic $\phi(x)$: (a) it is a bounded monotonic odd function, and $\phi'(x)$ is an even function, monotonic for $x > 0$; (b) $\phi'(x)$ has its maximum at the operating point $x = 0$ and decreases monotonically from the maximum on both sides of zero.

As a consequence of the assumptions if $\phi'(0) > 1/kr$, there are two values $\pm x_1$ ($x_1 > 0$) such that

$$\phi'(x_1) = \phi'(-x_1) = 1/kr.$$

When $\phi'(x) = 1/kr$ both derivatives $\dot{x},\dot{y}$ become infinite and consequently the values of x and those of y can change abruptly. The jump conditions yield here that the voltage across the condenser

$$V = \frac{1}{C}\int \frac{x}{kr}\,dt = Ly - \frac{x}{k},$$

and the current in the inductance

$$I = \int y\,dt = \phi(x) - \frac{x}{kr}$$

must both remain constant. Hence if x_1,y_1 and x_2,y_2 are the values

of x and y before and after the jump we must have:

$$\text{(4)} \qquad \begin{cases} \phi(x_1) - \dfrac{x_1}{kr} = \phi(x_2) - \dfrac{x_2}{kr}, \\ Ly_1 - \dfrac{x_1}{k} = Ly_2 - \dfrac{x_2}{k}. \end{cases}$$

When these conditions are fulfilled, the jump proceeds in the following way. The current $i = x/kr$ undergoes a jump and therefore the plate current also changes abruptly. According to the first condition (4) the change in the plate current I_a must be exactly equal to the change in the current i, so that the current I may remain constant. On the other hand, when the current i jumps, the voltage across r jumps, and the voltage across L likewise changes abruptly. According to the second condition (4) the voltage jump across r must be exactly equal to the jump of the voltage across L, and as a result the voltage across C remains constant.

As we shall see the system (3) plus the conditions (4) for the jump will enable us to determine the behavior of the system throughout the whole plane.

Let us set $\phi'(0) = g$ (for simplicity we write g for the standard g_m). It is here $\neq 0$ and we will suppose $g \neq 1/kr$. The origin is then the only singular point in the region where the system (3) is analytic, i.e. where $\phi'(x) \neq 1/kr$. Thus if

$$\text{(5)} \qquad \Phi(x) = \frac{1}{\phi'(x) - \dfrac{1}{kr}}$$

we will have the series expansion

$$\text{(6)} \qquad \Phi(x) = \Phi(0) + x\Phi'(0) + \cdots = \frac{1}{g - \dfrac{1}{kr}} + \alpha x + \cdots$$

Upon substituting in (3) and keeping only the first degree terms there results the system of the first approximation for the origin

$$\text{(7)} \qquad \dot{x} = \frac{y}{g - \dfrac{1}{kr}}, \qquad \dot{y} = \frac{x}{krLC} + \frac{1}{kL}\frac{y}{\left(g - \dfrac{1}{kr}\right)}.$$

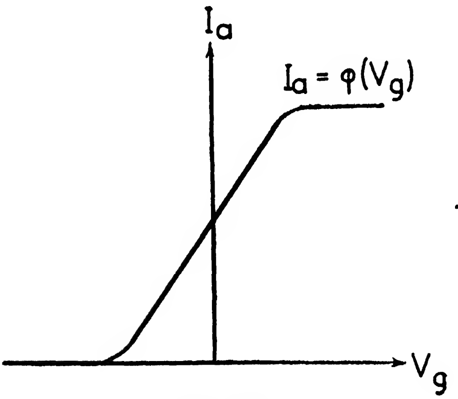
If we set

$$\rho = \frac{r}{krg - 1},$$

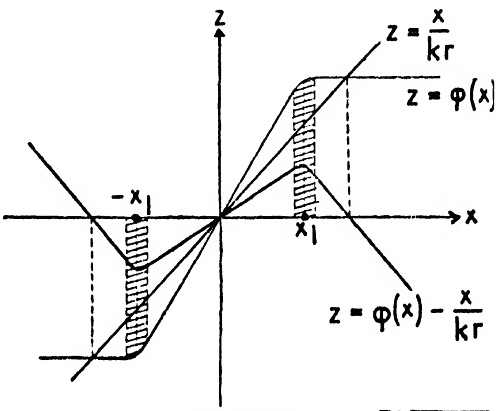
then the characteristic equation is found to be

$$\lambda^2 - \frac{\rho}{L}\lambda - \frac{\rho}{rLC} = 0. \tag{8}$$

When $\rho > 0$, i.e. when $krg > 1$, the roots are real and of opposite sign and the origin is a saddle point. When $krg < 1$ all the coefficients of (8) are positive and so we have a stable node or focus. The roots are real if $\rho^2 rC > -4\rho L$, and we have then a stable node; they are complex when the inequality is reversed and we have then a stable focus. In particular it is a stable node when L is small and a stable focus when L is large. However when $krg < 1$ the expression $\phi'(x) - 1/kr$ never vanishes. Hence it is only when $krg > 1$, i.e. only when



[Fig. 232.]



[Fig. 233.]

the origin is unstable, that jumps may occur. Let us suppose explicitly $krg > 1$, so that the origin is a saddle point.

Now (3) operates in three regions: Ω_1 to the left of $x = -x_1$, Ω_2 between the lines $x = \pm x_1$ and Ω_3 to the right of $x = x_1$. Since Ω_1 and Ω_3 contain no singular point, and Ω_2 only a saddle point, none of the three regions may contain a (continuous) limit-cycle. Hence the system does not admit continuous oscillations. Any oscillation that may occur must be discontinuous. We shall prove that under certain simplifying assumptions such oscillations do exist.

Let us then assume that the characteristic of the tube is linear over almost the entire region from zero to maximum (Fig. 232), so that in Ω_2 (Fig. 233) we almost always have

$$\phi'(x) = \text{constant} = g > \frac{1}{kr},$$

while $\phi'(x) = 0$ outside of Ω_2. The characteristic is curvilinear only over two small strips of width Δx, containing $x = x_1$ and $x = -x_1$ (shaded regions in Fig. 233). Within each strip the system is nonlinear, but it behaves like a linear system between the strips and outside them. These two regions must however be discussed separately.

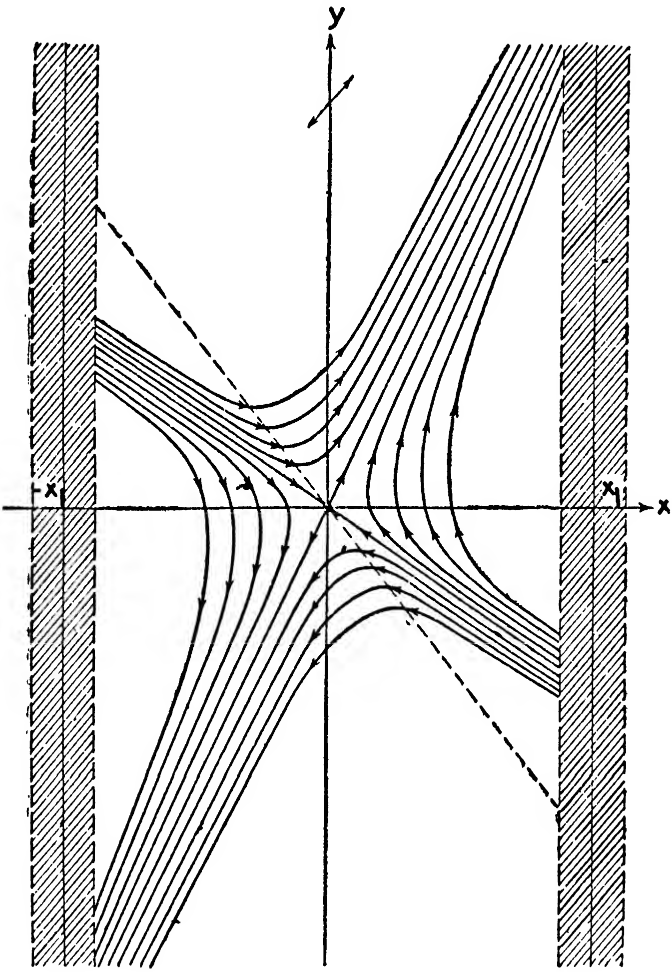


Fig. 234.

Region between the strips. The corresponding equation is (7) and the origin is a saddle point. The paths in this region are concentric hyperbolas and the asymptotes to the paths have the slopes

$$\frac{1}{krLC}\left(\frac{rC}{2} \pm \sqrt{\left(\frac{rC}{2}\right)^2 + LC(krg - 1)}\right).$$

The direction of motion along the paths is to be determined by reference to the initial system (7). Fig. 234 gives the portrait of the paths in the region.

Region exterior to the strips. The basic system is again (7) but with $g = 0$, or

$$\dot{x} = -kry, \qquad \dot{y} = \frac{x}{krLC} - \frac{r}{L}y \tag{9}$$

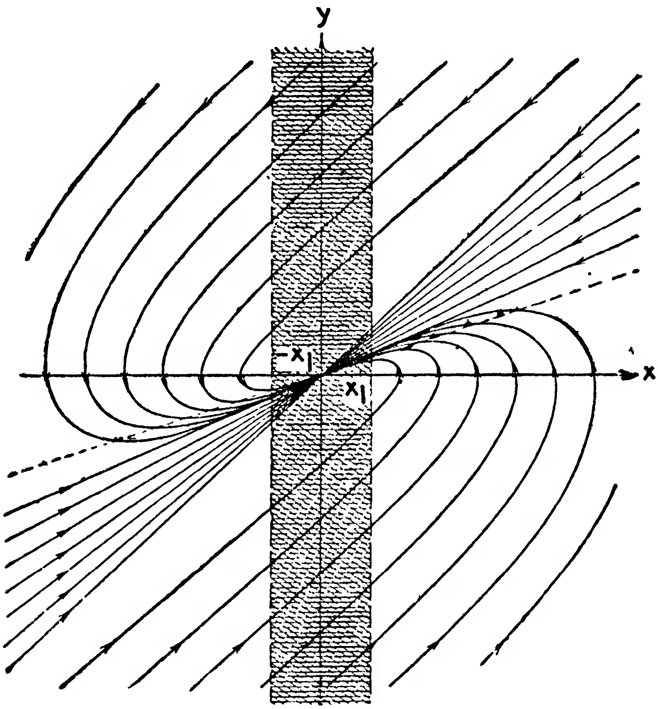


FIG. 235.

with characteristic equation

$$LC\lambda^2 + rC\lambda + 1 = 0$$

and characteristic roots

$$\lambda = \frac{-rC \pm \sqrt{r^2C^2 - 4LC}}{2LC}.$$

In the whole plane the paths of (9) are those of a linear system with a node when $r > 2\sqrt{L/C}$ (Fig. 235) or with a focus when $r < 2\sqrt{L/C}$ (Fig. 236). In both figures the shaded part represents the portion between the strips of Fig. 234, and only the paths in the unshaded part

are to be considered. Here also the direction of motion is obtained by reference to the initial system (9). Upon examining the direction of motions on the paths we see that whatever its initial position the representative point must sometime reach one of the verticals $x = \pm x_1$. On these verticals $\phi'(x) - 1/kr = 0$, hence both $\dot{x}$ and $\dot{y}$ become infinite and a jump must occur. It will proceed in accordance with the relations (4). The first asserts that $z(x) = \phi(x) - (x/kr)$

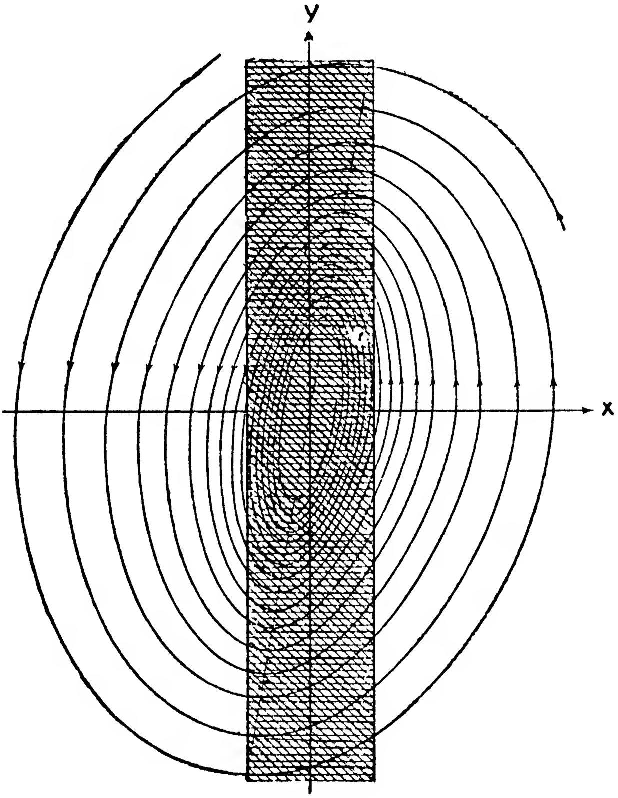


FIG. 236.

must not change. The graph of Fig. 237 shows that to x_1 there corresponds a unique value $-x_2$ $(x_2 > 0)$ such that $z(x_1) = z(-x_2)$. Since $z(-x) = -z(x)$, similarly $z(-x_1) = z(x_2)$ so that $-x_1$ will correspond similarly to x_2. As for y the second jump condition (4), asserts that (x_1,y_1) goes to a point on the line of slope $1/Lk$ through (x_1,y_1). That is to say the segment from the new position to the old has the fixed slope $1/Lk$. Thus to construct the new position one must merely draw a parallel to this direction, and if the position before the

jump is on $x = +x_1$ [on $x = -x_1$] find its intersection with $x = -x_2$ [with $x = +x_2$] which will be the position after the jump.

To simplify matters still further let us assume that the characteristic consists exclusively of rectilinear portions (Fig. 238). It is then

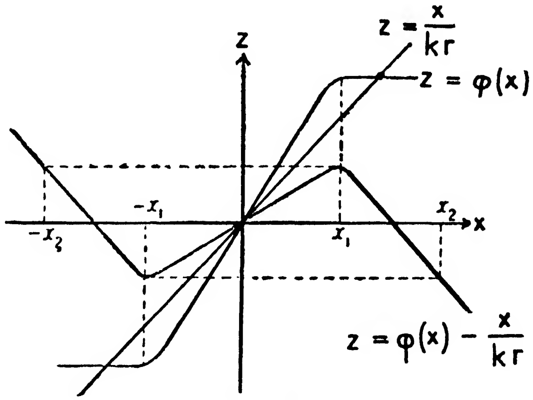


FIG. 237.

easy to find the relations between the parameters of the tube and x_1 and x_2. In fact, as is indicated in Fig. 238, $x'/kr = i' = I_s/2$. On the other hand, $x_1/kr = V_s/2kr = i_1$, where V_s is the so called saturation voltage of the tube I, i.e. the voltage that must be applied to the grid of the tube to make the plate current increase from

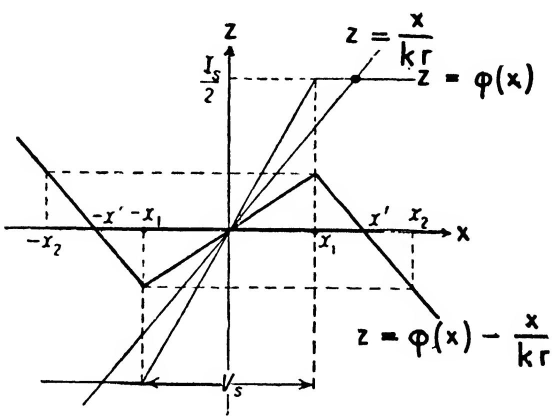


FIG. 238.

zero to saturation. Then

$$x_2/kr = i_2 = i_1 + 2(i' - i_1)$$

and hence

$$x_2 = \frac{V_s}{2} + 2kr\left(\frac{I_s}{2} - \frac{V_s}{2kr}\right).$$

Consequently

$$x_1 = \frac{V_s}{2}, \qquad x_2 = \frac{V_s}{2} - V_s + krI_s = V_s\left(\frac{krI_s}{V_s} - \frac{1}{2}\right)$$

and finally

(10) $$X = x_1 + x_2 = krI_s.$$

The details of the construction are of course affected by the size of the inductance L. In particular certain approximate computations

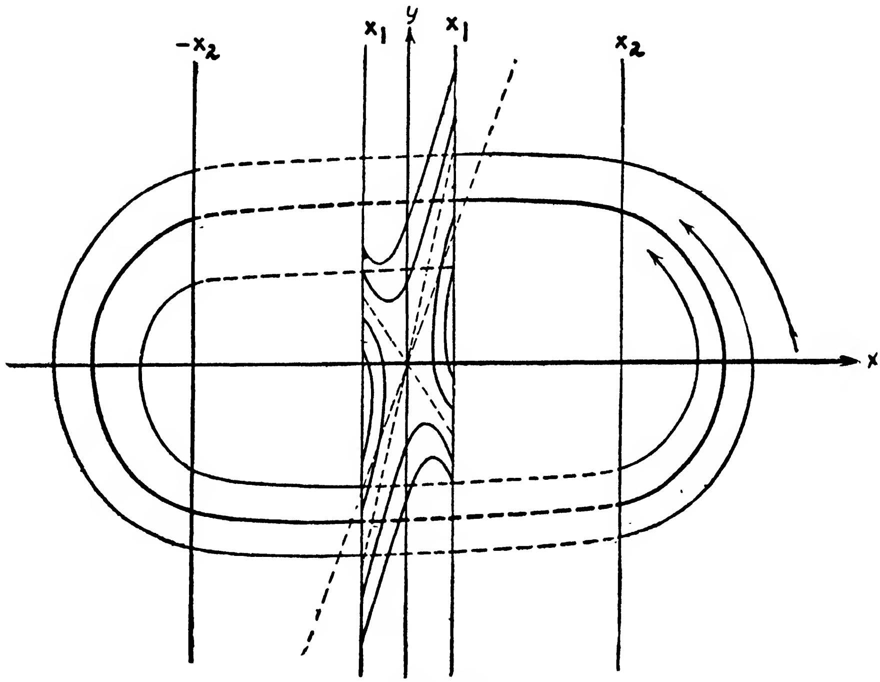


FIG. 239.

proceed differently for L very large and L very small and so they are best discussed separately.

L very large. The segments across the jumps, whose slopes are $1/kL$, are almost horizontal. Since $r \ll 2\sqrt{L/C}$ the situation of Fig. 236 prevails and the paths are as drawn in Fig. 239 where the connecting segments are shown in dotted lines. Considerations of continuity show that there must exist a pair of portions of the spirals whose extremities are closed by a jump thus producing a "closed" path (indicated in the figure with heavy lines) to which there corresponds a periodic motion. Upon examining the motions inside and

outside of this closed path one sees that it is stable. In fact, if the representative point moves along one of the internal curls of the spiral, it "jumps out" far outside and as a result the oscillations grow. Conversely, in the far removed regions, the system "jumps out" and remains after the jump inside of the spiral, within which it was situated before the jump, and as a result the oscillations dampen out. For some intermediary position there is compensation and steady discontinuous oscillations are produced. Assuming nearly sinusoidal oscillations we can compute approximately their amplitude. Let δ be the logarithmic decrement of the linear oscillations, Δy_1 the loss in y through a half oscillation, Δy_2 the variation in y through a jump. In the periodic motion $\Delta y_1 = \Delta y_2$. On the other hand

$$\Delta y_1 = y_0(1 - e^{-\frac{\delta}{2}}), \qquad \Delta y_2 = \frac{1}{kL}(x_1 + x_2) = \frac{X}{kL} = \frac{rI_s}{L}.$$

From this follows for the steady state amplitude E of the voltage at the terminals of the inductance

$$E = Ly_0 = \frac{X}{k(1 - e^{-\frac{\delta}{2}})} = \frac{rI_s}{(1 - e^{-\frac{\delta}{2}})}.$$

Finally, when $kr(I_s/V_s) \gg 1$ and $\delta \ll 1$ this approximate formula can be simplified further as

$$E = \frac{2rI_s}{\delta} = \frac{2I_s}{\pi}\sqrt{\frac{L}{C}}.$$

It is interesting to note, that when δ is small, $y_0 = 2I_s/\pi\sqrt{LC}$, i.e. the "amplitude" y_0 decreases when L increases. We can calculate approximately the period of self-oscillations corresponding to the case under investigation. Along the spirals the representative point would pass, without jumping, from y_0 to y_0', in a time

$$T = \frac{2\pi}{\sqrt{\omega_0^2 - h^2}},$$

where $\omega_0^2 = 1/LC$ and $h = r/2L$. It takes however less time since the jump is "instantaneous." The time saved may be calculated in the following way. The region across which the system jumps, would be crossed by a linear system with almost constant velocity. This velocity is found approximately by means of (9) as $\dot{x} = -kry_0$. Consequently, the time necessary to change x twice from x_1 to x_2, i.e.

to vary x by the magnitude X is approximately the correction for the period:

$$\tau = 2\frac{X}{kry_0} = \frac{2L}{r}(1 - e^{-\frac{\delta}{2}}).$$

Thus the correction to the period is no longer a magnitude of second order with respect to δ, as in the case of an ordinary vacuum tube generator. It is quite important that when the damping decreases, i.e. when δ decreases and L/r increases, the correction to the period does not tend to zero but to a certain finite value which can be easily determined. Namely, when the damping is small, i.e. when $\delta \ll 1$, we obtain:

$$\tau = \frac{2L}{r}\cdot\frac{\delta}{2} = \frac{L}{r}\cdot\frac{rT_0}{2L} = \frac{T_0}{2},$$

where T_0 is the period of oscillations in a frictionless linear system. When L is large, i.e. when damping is small, $T_0 = T$ (T = duration of the "period" of damped oscillations) and consequently, the period of self-oscillations is approximately $T_1 = T_0/2$ when L is large.

L very small. The segments across the jump are now almost vertical. Since $r \gg 2\sqrt{L/C}$, the situation is as in Fig. 235. The equation of the paths is (Chap. I, §6)

$$(y - q_1x)^{q_1} = D(y - q_2x)^{q_2}$$

where

$$q_1 = -h + \sqrt{h^2 - \omega_0^2}, \qquad q_2 = -h - \sqrt{h^2 - \omega_0^2},$$

$$h = -\frac{1}{2kL}, \qquad \omega_0^2 = \frac{1}{k^2r^2LC}.$$

If h^2 is much larger than ω_0^2, we have approximately

$$q_1 = -\frac{\omega_0^2}{2h} = \frac{1}{kr^2C}, \qquad q_2 = -2h = \frac{1}{kL},$$

where q_1 and q_2 represent the slopes of the "integral lines," corresponding to $D = 0$ and $D = \infty$. A detailed construction shows that in this case also there must exist a stable periodic motion, consisting of two motions one with finite velocity, and the other of two jumps. Here again we have discontinuous oscillations. Their amplitude can be determined at once since x varies from x_2 to $-x_1$ and from $-x_2$ to x_1. The variation of y can be determined as in the general case from conditions (4) for the jump. The determination of the period is greatly

simplified when L is very small, more precisely when $h^2 \gg \omega_0^2$. In this case the "direction" of the jump and the direction of the path $y = q_2x$ practically coincide (Fig. 240). We can assume that the motion has a finite velocity between the points a and b, and between the points c and d of the path $y = -2hx = x/kL$ (the system jumps from b to c and from d to a). Since this motion follows equation (3) where $\phi'(x) = 0$, i.e.

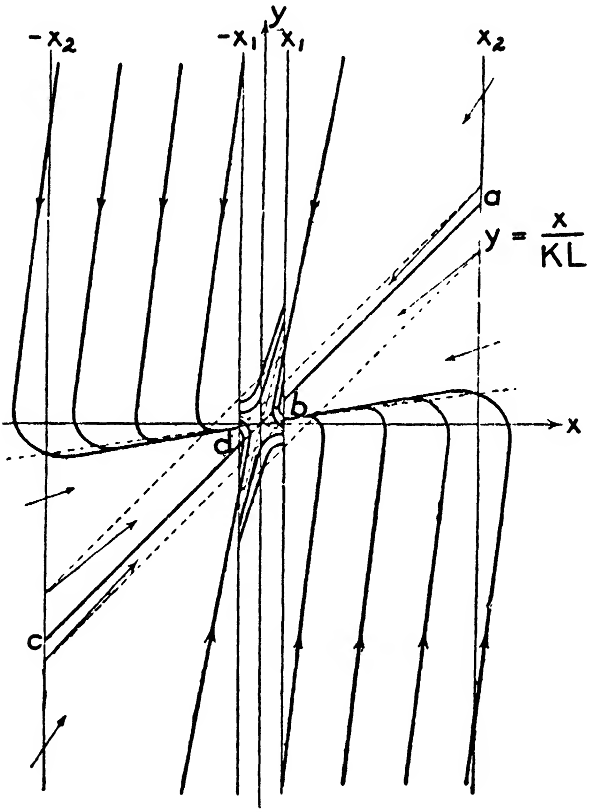


FIG. 240.

the equation of a linear system, then

$$\dot{x} = -kry.$$

On the other hand, we have $y = x/kL$ and, consequently,

$$dt = -\frac{L}{r}\frac{dx}{x}.$$

Hence the half period is

$$\frac{T}{2} = -\int_{x_2}^{x_1} \frac{L\,dx}{rx} = \frac{L}{r}\log\frac{x_2}{x_1} = \frac{L}{r}\log 2\left(kr\frac{I_s}{V_s} - \frac{1}{2}\right).$$

It is interesting to note that in this case the capacitance C does not appreciably affect the period, for if $h^2 \gg \omega_0^2$, then $C \gg 4L/r^2$. A sufficiently large capacitance does not affect the period of self oscillations.

While we have only discussed, for convenience, the extreme values of L, the intermediary values are likewise amenable to treatment but with somewhat heavier mathematical machinery. Figs. 241a, b, c, d, are

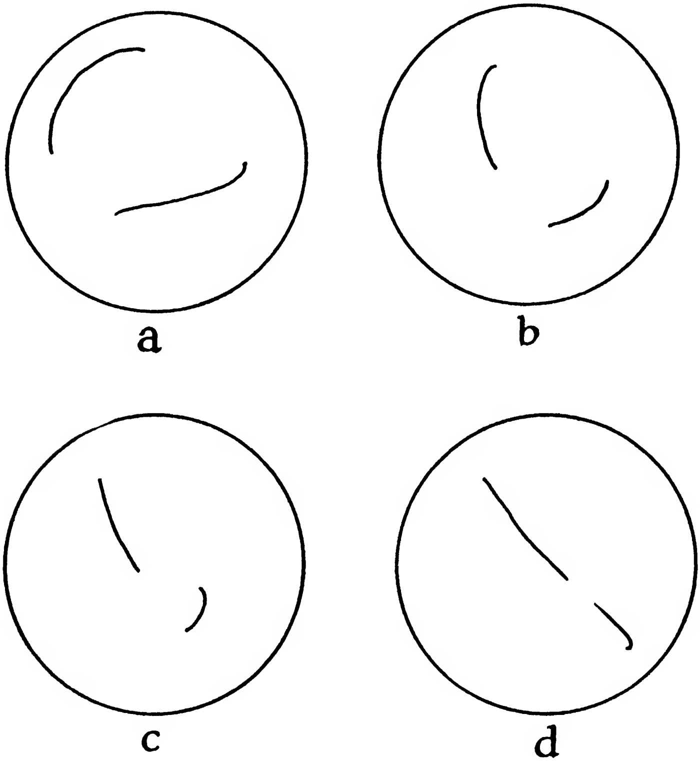


FIG. 241.

pictures of the phase plane for decreasing values of L. These tracings were obtained with the aid of a cathode ray oscilloscope. The character of the periodic process represented in these photographs agree with our theoretical investigation. The lack of symmetry in the tracings is due to the difficulties one has in eliminating the asymmetry in the circuits themselves.

§2. SYSTEMS WITH TWO DEGREES OF FREEDOM DESCRIBED BY TWO EQUATIONS OF THE FIRST ORDER

A system with two degrees of freedom is normally represented by two equations of the second order. Should both degenerate to the first

order the situation is as described in the title with possible discontinuous oscillations. A few examples will serve as illustrations.

1. The Fruehauf circuit. In this circuit (Fig. 242) a substantial role is played by the circumstance that the tubes themselves are points of the closed circuits under discussion. Therefore, one has to take into account the distribution of voltage between the tubes and the external resistances. This leads to the assumption that the plate voltage is a function of the plate current, i.e., to take into account the plate reaction, which in this arrangement, plays the predominant role. Therefore, we shall assume here that the plate current is not only a function of the grid voltage but also of the plate voltage. Using the

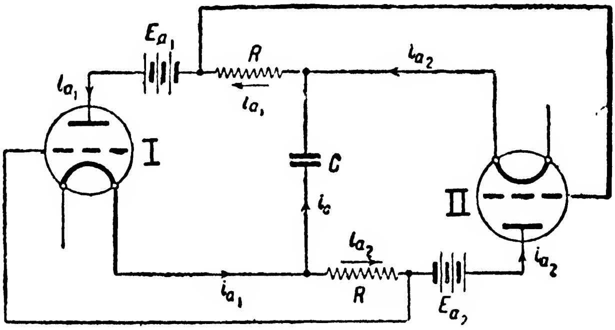


FIG. 242.

notations of Fig. 242 and neglecting the grid currents (assuming that the two tubes are identical) we have:

$$
\begin{aligned}
i_{a_1} &= f(e_{g_1} + De_{a_1}), \qquad & e_{g_1} &= E_{g_1} - Ri_{a_2} \\
i_{a_2} &= f(e_{g_2} + De_{a_2}), \qquad & e_{g_2} &= E_{g_2} - Ri_{a_1}
\end{aligned}
$$

$$e_{a_1} = E_{a_1} - Ri_{a_1} - \frac{1}{C}\int_0^t (i_{a_1} - i_{a_2})\,dt$$

$$e_{a_2} = E_{a_2} - Ri_{a_2} + \frac{1}{C}\int_0^t (i_{a_1} - i_{a_2})\,dt$$

where e_{g_1} and e_{a_1}, e_{g_2} and e_{a_2} represent respectively the grid and the plate voltages of the tubes I and II and $1/D$ is the amplification factor. We shall assume that the characteristic of the tube $i_a = f(e_{st}) = f(e_g + De_a)$ is such that the inverse function $e_{st} = \phi(i_a)$ is single-valued. If we set

$$i_{a_1}R = x, \qquad i_{a_2}R = y, \qquad \phi\left(\frac{u}{R}\right) = \psi(u),$$

we have

$$
(11)\quad \begin{cases} E_{g_1} + DE_{a_1} - y - Dx - \dfrac{D}{CR}\displaystyle\int_0^t (x - y)\,dt = \phi(i_{a_1}) = \psi(x), \\ E_{g_2} + DE_{a_2} - x - Dy + \dfrac{D}{CR}\displaystyle\int_0^t (x - y)\,dt = \phi(i_{a_2}) = \psi(y). \end{cases}
$$

These relations may also be written

$$
(12)\qquad E_{g_1} + DE_{a_1} - y - Dx - \frac{D}{CR}\int_0^t (x - y)\,dt = \psi(x),
$$

$$
(13)\quad E_{g_1} + E_{g_2} + D(E_{a_1} + E_{a_2}) - (y + x)(D + 1) = \psi(x) + \psi(y).
$$

The last relation can in fact be read off from the circuit diagram. It shows that the representative point (x,y) must remain on the curve (13) which we denote by Γ. We see here that we are in fact in presence of simple degeneracy. This is brought out more clearly in a moment.

Differentiating (11) yields the system

$$
(14)\quad \begin{cases} \dot{x} = \dfrac{-D}{CR}\,\dfrac{x - y}{\Delta(x,y)}\,(\psi'(y) + D + 1) \\ \dot{y} = \dfrac{D}{CR}\cdot\dfrac{x - y}{\Delta(x,y)}\,(\psi'(x) + D + 1) \\ \Delta(x,y) = (\psi'(x) + D)(\psi'(y) + D) - 1. \end{cases}
$$

Since x and y are related by (13) we may imagine (13) solved for y in terms of x. Then upon substituting in the first equation (14) there would result an equation of the form $\dot{x} = f(x)$. Thus essentially a non-degenerate system of two differential equations of order two has degenerated to a single one of order one—this is the simple degeneracy already mentioned. The process just indicated cannot generally be carried out effectively as (13) is not generally solvable in closed form for y. For this reason it is just as well to deal directly with (14) remembering however that the representative point $M(x,y)$ is on the fixed curve Γ. We shall see that here also discontinuous oscillations can arise.

Let Δ denote the curve $\Delta = 0$. As the point M describes Γ and reaches Δ at $M_0(x_0,y_0)$ both $\dot{x}$ and $\dot{y}$ become infinite and a jump may occur to $M_1(x_1,y_1)$. Upon expressing the constancy of the capacitor voltage as the current undergoes a jump one obtains the relations

$$
(15)\quad \begin{cases} y_0 + \psi(x_0) + Dx_0 = y_1 + \psi(x_1) + Dx_1, \\ x_0 + \psi(y_0) + Dy_0 = x_1 + \psi(y_1) + Dy_1. \end{cases}
$$

From this follows that $(D + 1)(x + y) + \psi(x) + \psi(y)$ has the same value at M_1 as at M_0, and hence that M remains on Γ as one should expect. Assuming also that M_0 describes Δ, the relations (15) assign to each position of M_0 one of M_1 and their totality generates the curve B. The general disposition of the curves is in accordance with Fig. 244. Under the usual restrictions on the characteristic ψ both curves are closed.

To obtain further insight into the question it is necessary to adopt a suitable analytical expression for the characteristic. Let us assume that it is approximated by

$$i_a = \frac{I_s}{2} + \frac{I_s}{\pi}\arctan\frac{3}{v_s}(e_{st} - v_s),$$

where I_s is the saturation current, v_s the saturation voltage, and $e_{st} = e_g + De_a$ the directing voltage. Solving for v_s and using the notations introduced earlier, we obtain:

$$\psi(x) = v_s - \frac{v_s}{3}\cot\frac{\pi x}{RI_s}, \qquad \psi(y) = v_s - \frac{v_s}{3}\cot\frac{\pi y}{RI_s}.$$

For the sake of convenience let us pass to dimensionless magnitudes by introducing the new variables

$$\frac{\pi}{RI_s}x = u, \qquad \frac{\pi}{RI_s}y = v, \qquad \frac{D}{RC}t = \tau.$$

As a consequence (14) is replaced by

$$\text{(16)} \qquad \begin{cases} \dfrac{du}{d\tau} = -(u - v)\dfrac{D + 1 + \dfrac{a}{\sin^2 v}}{\Delta(u,v)}, \\[2ex] \dfrac{dv}{d\tau} = +(u - v)\dfrac{D + 1 + \dfrac{a}{\sin^2 u}}{\Delta(u,v)} \end{cases}$$

with the relations

$$(D + 1)(u + v) - a(\cot u + \cot v) = c',$$
$$\Delta(u,v) = \left(D + \frac{a}{\sin^2 u}\right)\left(D + \frac{a}{\sin^2 v}\right) - 1.$$

For the jump we have

$$v_1 - a \cot u_1 + Du_1 = v_0 - a \cot u_0 + Du_0,$$
$$u_1 - a \cot v_1 + Dv_1 = u_0 - a \cot v_0 + Dv_0.$$

The constants a and c' are given by

$$a = \frac{\pi v_s}{3RI_s}, \qquad c' = \frac{\pi}{RI_s}(E_{g_1} + E_{g_2} + D(E_{a_1} + E_{a_2}) - 2v_s)$$

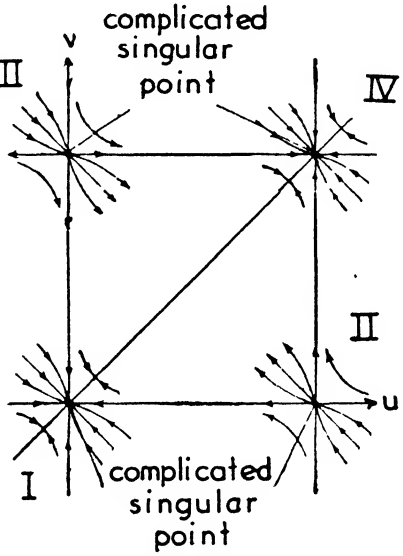


FIG. 243.

The singular points of the system (16) are determined by

$$(u - v) \sin^2 u \,((D + 1) \sin^2 v + a) = 0,$$
$$(u - v) \sin^2 v \,((D + 1) \sin^2 u + a) = 0.$$

These equations give the coordinates of the four following singular points (Fig. 243):

$$\text{I. } (u = 0, v = 0); \qquad \text{III. } (u = 0, v = \pi);$$
$$\text{II. } (u = \pi, v = 0); \qquad \text{IV. } (u = \pi, v = \pi).$$

The investigation of these singular points shows that they have a complicated character being in fact of the third order (Chap. V, §4). Since the plate current can vary within the limits $0 \leqslant i_a \leqslant I_s$, only the portion of the phase plane defined by the inequalities:

$$0 \leqslant u \leqslant \pi; \qquad 0 \leqslant v \leqslant \pi$$

has physical meaning. In this region the paths behave in the vicinity of the singular points as if the points II and III were nodes and the points I and IV saddle points. Let us investigate the curves Δ, B and the paths. In the plane the curve Δ is a closed curve symmetrical with respect to the lines:

$$u = v; \qquad u + v = \pi; \qquad u = \pi/2; \qquad v = \pi/2.$$

The curve B constructed graphically is also symmetrical with respect to the lines $u = v$ and $u + v = \pi$. The paths consist of curves symmetrical with respect to the line $u + v = \pi$ and each is also symmetrical with respect to the line $u = v$. The motion of the

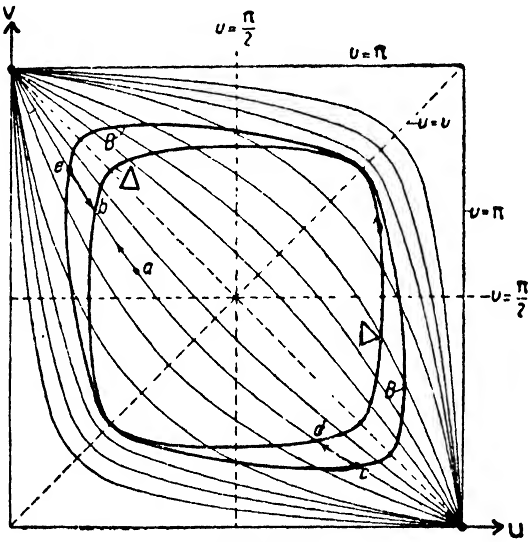


FIG. 244.

representative point M has the following character: starting from an arbitrary point, a for example, it moves along the path and comes to the point b of the curve Δ; from there the representative point jumps to the point c of the curve B. Then, continuing along the same path it returns to the point d on the curve Δ, and from there "jumps" to the point e. Then it follows the path up to the point b, etc.

Let us determine the values of the parameter for which there arise discontinuous oscillations. They can obviously take place if: (1) the curve Δ has real branches, (2) the path Γ intersects Δ. One can show that Δ will have real branches if $a < 1 - D$. The second condition will be satisfied if:

$$2(D+1)\arcsin\sqrt{\frac{a}{1-D}} - 2a\sqrt{\frac{1-D-a}{a}} < c'$$
$$< 2(D+1)\left(\pi\text{-arcsin}\sqrt{\frac{a}{1-D}}\right) + 2a\sqrt{\frac{1-D-a}{a}}$$

where the values of arcsin correspond to the first quadrant and the radicals are taken positively. The calculation of the period of oscillations in the motion along an arbitrary path meets with appreciable mathematical difficulties. For this reason we shall merely carry it out for the path which is the diagonal line $u + v = \pi$. Then

$$\frac{du}{d\tau} = \frac{-dv}{d\tau} = -(2u - \pi)\frac{D + 1 + \dfrac{a}{\sin^2 u}}{\left(D + \dfrac{a}{\sin^2 u}\right)^2 - 1},$$

and therefore the period θ is

$$\theta = 2\int_{u_0}^{u_1}\frac{1-D}{\pi - 2u}\,du + 2a\int_{u_2}^{u_1}\frac{du}{(2u-\pi)\sin^2 u}$$

where u_0 and u_1 are respectively the coordinates of the intersection points of $u + v = \pi$ and the curves Δ and B. Since $a \ll 1$ one may neglect the second integral. Passing then to the ordinary time scale we finally obtain for the period the following approximate formula:

$$T = RC\,\frac{1-D}{D}\log\frac{\pi - 2u_0}{\pi - 2u_1}$$

2. Amplifier with RC-tuned feedback. This circuit of Haegner does not give rise to discontinuous oscillations and is only discussed here with our usual simplifications for a later purpose. It represents at all events a twice degenerate system with two degrees of freedom. In fact, if we neglect the plate reaction and the grid current and use the notations for the currents and voltages described in Fig. 245, we can easily obtain the two following differential equations of the first order:

$$(17)\quad \begin{cases} \dot{i}_1 = -\dfrac{i_1}{rC_1} + \dfrac{i_2}{rC_2}, \\[2ex] \dot{i}_2 = -\dfrac{R + r - Rrk\phi'(kri_1)}{RrC_2}\,i_2 + \dfrac{R - Rrk\phi'(kri_1)}{RrC_1}\,\dot{i}_1. \end{cases}$$

We can see at once that in the case under investigation $\dot{i}_1$ and $\dot{i}_2$ remain finite when i_1 and i_2 are finite and that consequently, the currents can have no jumps. On the other hand, for the circuit under consideration, this system possesses only one singular point: $i_1 = 0$, $i_2 = 0$. Setting as before $\phi'(0) = g > 0$, the characteristic equation for the origin reduces to

$$\gamma^2 + \left(\frac{C_2}{C_1} + \frac{R + r - Rrkg}{R}\right)\frac{1}{rC_2}\gamma + \frac{1}{RrC_1C_2} = 0.$$

This equation shows that the origin cannot be a saddle-point but only

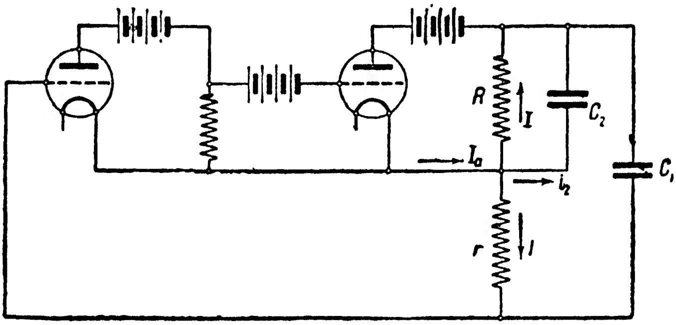


FIG. 245.

a node or a focus. It will be unstable when and only when

$$\frac{Rrkg - (R + r)}{R} > \frac{C_2}{C_1}, \tag{18}$$

If this inequality is "weak" we have an unstable focus, if the inequality is "strong" we have an unstable node. The infinity in this case is very unstable. In fact, at high voltages, the tube reaches saturation and ceases to work. Therefore in far removed regions, the scheme behaves as a linear system possessing a stable node. Consequently, the paths will all proceed from infinity into the region of finite i_1 and i_2. If the unstable singular point is situated at the origin, there exists at least one stable limit-cycle. If the characteristic decreases monotonically with a steep slope, this stable limit-cycle will be unique. To find the limit-cycle we can apply the isocline method, and it is shown in Fig. 246. In this scheme the oscillations are not only continuous but also similar to sinusoidal oscillations, since the limit-cycle is very similar to an ellipse. When the excitation is stronger (the inequality (18) is "stronger"), the shape of the limit-cycle is deformed and the oscil-

lations diverge more and more from sinusoidal oscillations. They remain, however, always continuous.

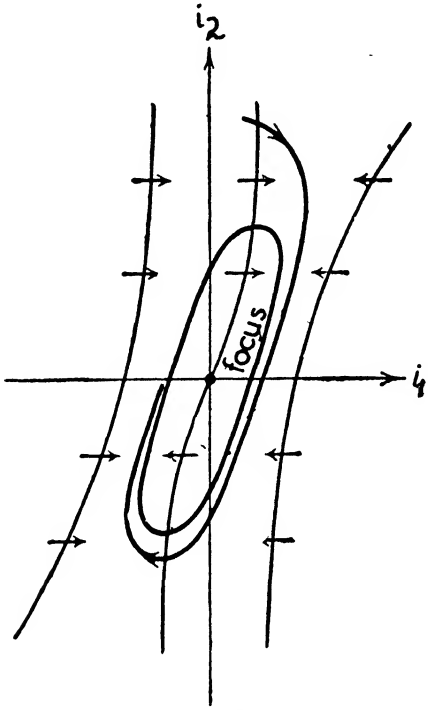


FIG. 246.

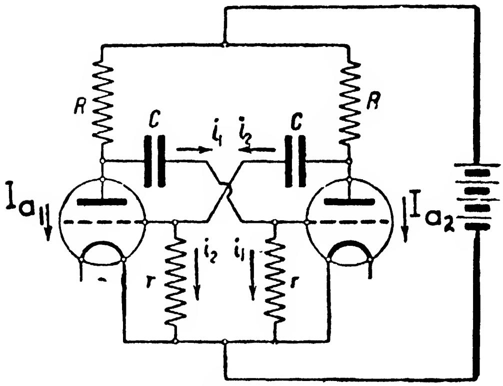


FIG. 247.

3. Multivibrator. In the circuit of Abraham and Bloch, known under the name of multivibrator, discontinuous oscillations can take place. In this scheme (Fig. 247) one neglects the grid current and the plate reaction and one assumes the circuit symmetrical. In the notations of the sketch we have the system:

$$(19)\qquad \begin{cases} \dot{i}_1 = \dfrac{(R+r)\dfrac{i_1}{C} - rR\phi'(ri_2)\dfrac{i_2}{C}}{r^2R^2\phi'(ri_1)\phi'(ri_2) - (R+r)^2}, \\[2ex] \dot{i}_2 = \dfrac{(R+r)\dfrac{i_2}{C} - rR\phi'(ri_1)\dfrac{i_1}{C}}{r^2R^2\phi'(ri_1)\phi'(ri_2) - (R+r)^2}. \end{cases}$$

This system is of the form

$$\dot{i}_1 = \frac{P(i_1,i_2)}{R(i_1,i_2)}, \qquad \dot{i}_2 = \frac{Q(i_1,i_2)}{R(i_1,i_2)}.$$

It has a single state of equilibrium at the origin $i_1 = i_2 = 0$. There are other common solutions of $P = Q = 0$ but where they occur likewise $R = 0$ and so they do not represent true states of equilibrium.

Let us look for continuous closed paths. They are the same for (19) as for

$$\frac{di_1}{P} = \frac{di_2}{Q}$$

or for

$$(20)\qquad \frac{di_1}{d\tau} = P, \qquad \frac{di_2}{d\tau} = Q.$$

Since

$$\frac{\partial P}{\partial i_1} + \frac{\partial Q}{\partial i_2} = \frac{2(R+r)}{C} > 0,$$

the Bendixson criterion asserts that (20) has no continuous limit cycles, and so the same holds regarding the system (19) itself.

The characteristic equation for the origin is

$$\gamma^2 + \frac{2(R+r)}{C((R+r)^2 - r^2R^2g^2)}\gamma + \frac{1}{C^2}\frac{1}{((R+r)^2 - r^2R^2g^2)} = 0,$$

where $g = \phi'(0) > 0$. The characteristic roots are easily seen to be always real and so the origin is either a node or a saddle point. When $R + r < rRg$ the roots are of opposite sign and we have a saddle point. When $R + r > rRg$ the roots are both negative and we have a stable node. However if $rRg = rR\phi'(0) > R + r$ then for some values i_1,i_2 we must have equality, that is to say we must have

$$r^2R^2\phi'(ri_1)\phi'(ri_2) = (R+r)^2$$

i.e. for certain values of i_1 and i_2 the denominator $R(i_1,i_2) = 0$, which means that $\dot{i}_1$ and $\dot{i}_2$ become infinite and the current in both circuits varies abruptly. Thus, on the one hand there are no continuous periodic solutions, and on the other the system can experience jumps and, consequently, discontinuous periodic solutions may possibly take place. The point $a(i_1',i_2')$ where the system jumps satisfies the relation:

$$R(i_1',i_2') = r^2R^2\phi'(ri_1')\phi'(ri_2') - (R + r)^2 = 0 \tag{21}$$

If $\phi'(0)rR > R + r$ and $\phi'(ri)$ decreases monotonically on both sides of zero, (21) represents a closed curve Γ surrounding the origin and

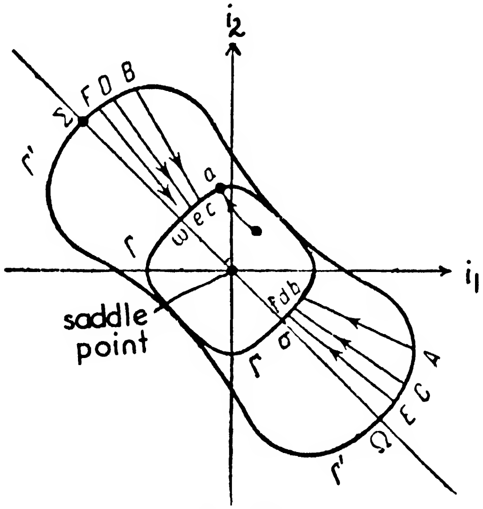


FIG. 248.

symmetrical with respect to the axes (Fig. 248). The point $A(i_1'',i_2'')$ to which the representative point jumps from the point a on the curve Γ is determined by the condition for the jump, according to which the voltages of the plates of the capacitor

$$V_1 = E - R\phi(ri_2) - (R + r)i_1, \qquad V_2 = E - R\phi(ri_1) - (R + r)i_2$$

do not change. Hence the relations

$$\begin{aligned} R\phi(ri_2') + (R + r)i_1' &= R\phi(ri_2'') + (R + r)i_1'' \\ R\phi(ri_1') + (R + r)i_2' &= R\phi(ri_1'') + (R + r)i_2'' \end{aligned}$$

Given the point a these conditions determine A uniquely. As a describes the curve Γ, A describes a curve Γ' (Fig. 248) representing the geometric locus of points to which the system jumps. Γ' is a closed curve surrounding Γ and it is likewise symmetrical with respect

to $i_1 = \pm i_2$. The representative point coming from any region in the phase plane reaches the curve Γ at the point a, for example. It proceeds then along Ab, Bc, Cd, etc., with jumps from a to A, from b to B, from c to C, etc. One can show that during this motion the system comes closer and closer to a periodic process consisting of two continuous motions, from Ω to σ and from Σ to ω, and of two jumps, from ω to Ω and from σ to Σ. This periodic motion continuously satisfies the symmetry relation $i_1 = -i_2$. The symmetry in the oscillations is the result of the symmetry in the circuit.

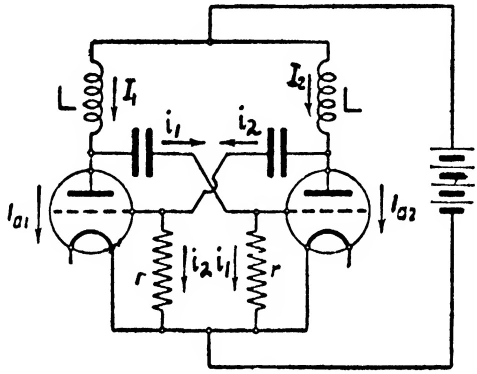


FIG. 249.

We could have assumed from the beginning that in the steady state the oscillations are symmetrical in i_1 and i_2 and therefore, in one of the equations (19) set $i = i_1 = -i_2$, hence $\phi'(ri_1) = \phi'(ri_2)$. As a result we would arrive at one equation of the first order

$$[rR\phi'(ri) - (R + r)]\dot{i} = \frac{i}{C}.$$

We have already investigated equations of this type (see Chap. IV, §11) when we studied oscillations in a system with one degree of freedom. Thus the differential equations of a symmetrical multivibrator are reducible to an equation of first order. This mode of treatment however does not extend to non-symmetrical initial conditions.

Observe that while the symmetry assumptions for the oscillation have substantially simplified the discussion, they were not necessary. In the general case, however, symmetrical schemes with two degrees of freedom lead to two equations of the second, and not of the first, order, i.e. to a very complicated mathematical problem.

The assumption of symmetry is quite well verified in practice if

the circuit itself is sufficiently symmetrical. This is the case in the so-called "multivibrator with inductance" (Fig. 249) utilized in the measurement of radio-frequencies. It leads to the following two equations:

$$rL\phi'(ri_2)\frac{di_2}{dt} + L\frac{di_1}{dt} + ri_1 + \frac{1}{C}\int i_1\,dt = E,$$

$$rL\phi'(ri_1)\frac{di_1}{dt} + L\frac{di_2}{dt} + ri_2 + \frac{1}{C}\int i_2\,dt = E.$$

If we assume that the oscillations in the circuit are symmetrical, i.e., that $i_1 = -i_2$, we find here

$$L[1 - r\phi'(ri)]\dot{i} + ri + \frac{1}{C}\int i\,dt = E.$$

This equation is completely equivalent to the system at the beginning of §1 with one degree of freedom, which includes L, C and r. Hence

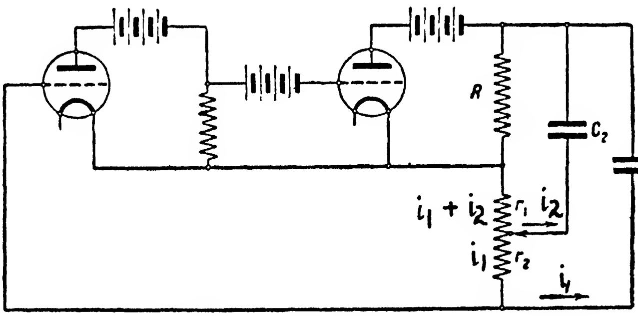


FIG. 250.

the conclusions obtained there are applicable here, save that one must make $k = 1$.

We have examined two schemes, one of which can experience only continuous and the other only discontinuous oscillations. There exist, however, schemes in which continuous oscillations change to discontinuous oscillations when a certain parameter varies. Take the "universal scheme" (Chap. V, §3) of Fig. 250. If we move the contact k along the resistance r, and the ratio $\beta = r_1/r$, $r = r_1 + r_2$, reaches a certain value, the node is transformed into a saddle point. At the same time continuous oscillations are transformed into discontinuous ones. This situation is clearly brought out by reference to the general equation of the scheme

$$\dot{i}_1 = \frac{-i_1}{(1-\beta)rC_1} + \frac{i_2}{(1-\beta)rC_2}$$

$$\dot{i}_2 = \frac{\{\beta r + R - krR\phi'(kr(i_1 + \beta i_2))\}\dfrac{i_1}{C_1} - \{r + R - krR\phi'(kr(i_1 + \beta i_2))\}\dfrac{i_2}{C_2}}{r(1-\beta)\{R + \beta r - \beta krR\phi'(kr(i_1 + \beta i_2))\}}$$

Upon writing down the characteristic equation it is found that when $\beta < \beta_{cr} = \frac{R}{r} \cdot \frac{1}{gkR - 1}$, the singular point represents a focus or a node and the system will have a limit-cycle. When $\beta > \beta_{cr}$, the singular

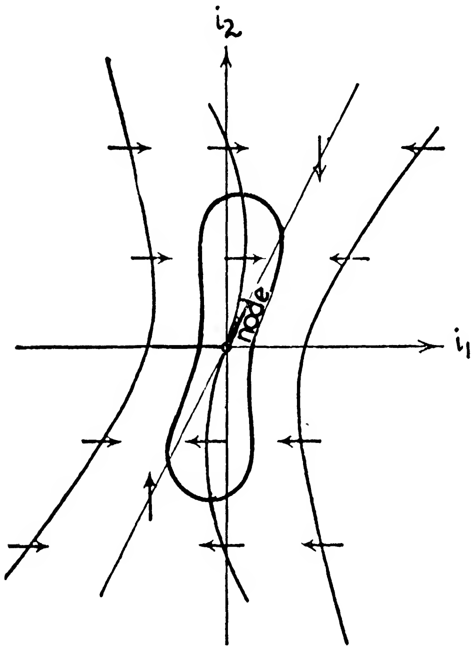


FIG. 251.

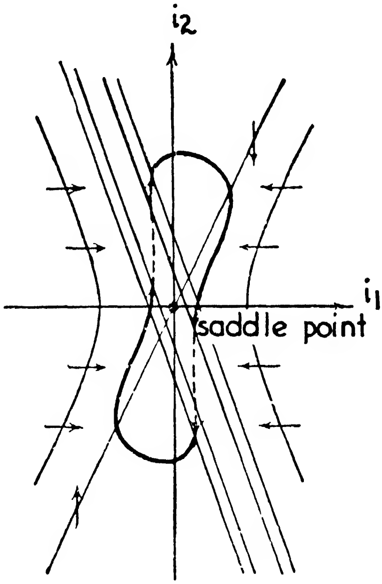


FIG. 252.

point becomes a saddle point, and the limit-cycle disappears. However as $\dot{i}_2$ can become infinite the current i_2 may undergo a jump and the system experience discontinuous oscillation analogous to those discussed in the previous paragraph. Figs. 251 and 252 describe the two cases.

§3. SMALL PARASITIC PARAMETERS AND STABILITY

In the systems reducible to two equations of the first order stability has always been governed by the nature of the roots of a characteristic equation of order two

$$\lambda S^2 + bS + c = 0. \tag{22}$$

Let us suppose however that as a consequence of disregarding certain

small parameters such as resistances, etc., the system reduces to a single equation of order one and that in (22) this makes $\lambda = 0$, reducing the equation to

$$bS + c = 0, \tag{23}$$

with A still a state of equilibrium. While b and c may well depend on λ, we will suppose, for the sake of simplicity, that they are polynomials $b(\lambda)$, $c(\lambda)$ in λ and that $b(0) \neq 0$. Now it may well happen that (23) asserts that the equilibrium is stable while (22) does not. It would certainly be important to know the "true" situation. Unfortunately in real physical systems the true situation is never known. All that one can do is to ascertain whether the verdict "stable" is to be modified if certain reasonably obvious stray impedances are taken into consideration.

To find out where we stand let us calculate approximate values for the characteristic roots of (22) under the assumption that λ is very small. Notice first that

$$\sqrt{b^2 - 4\lambda c} = b\left(1 - 4\frac{\lambda c}{b^2}\right)^{\frac{1}{2}} = b\left(1 - \frac{2\lambda c}{b^2} - \frac{2\lambda^2 c^2}{b^4} + \cdots\right)$$

where the expansion is according to the binomial theorem and valid for $|4\lambda c/b^2| < 1$, and certainly for λ very small. We may thus write

$$\sqrt{b^2 - 4\lambda c} = b - \frac{2\lambda c}{b} + 2k\lambda^2,$$

where $|k|$ is bounded as $\lambda \to 0$. As a consequence the roots are

$$S_{1,2} = \frac{-b \pm \sqrt{b^2 - 4\lambda c}}{2\lambda} = \frac{-b \pm \left(b - \dfrac{2\lambda c}{b} + 2k\lambda^2\right)}{2\lambda}$$

or finally if $S_0 = -c/b =$ the root of (23):

$$S_1 = -\frac{c}{b} + k\lambda = S_0 + k\lambda \doteq S_0,$$

$$S_2 = -\frac{b}{\lambda} + h \doteq -\frac{b}{\lambda}$$

where h is bounded as $\lambda \to 0$.

We may manifestly suppose $\lambda > 0$. Under these conditions we have:

Condition for stability according to (23) $S_0 = -c/b < 0$, i.e. b and c of same sign.

Conditions for stability according to (22): S_0 and $-b/\lambda < 0$, i.e. b and c both positive.

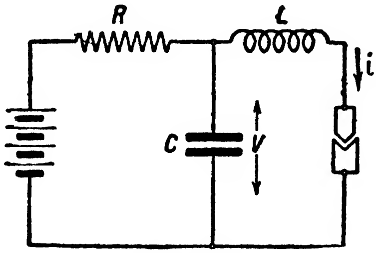


FIG. 253.

Consequently whenever b and c are both negative there is:

stability in accordance with (23), i.e. in accordance with the degenerate system caused by disregarding the small parasitic parameter λ;

instability in the form of a saddle point in accordance with (22), i.e. when the small parasitic parameter λ is taken into consideration.

We have then precisely the contradiction already mentioned. That the situation described above may actually arise in practice will now be shown by two examples.

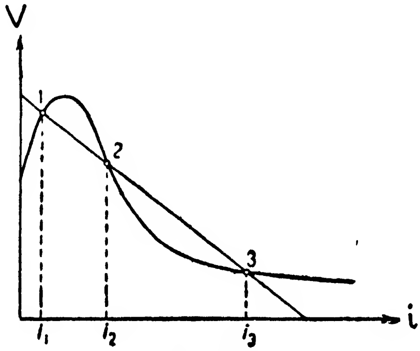


FIG. 254.

1. Consider a circuit containing a voltaic arc fed by a d.c. voltage supply (Fig. 253), and containing besides the battery and arc, a resistance R, an inductance L, and a capacitance C. Let the characteristic of the arc (dependence of V_b on i) be $V_b = \psi(i)$ (Fig. 254). The basic equations are

$$C\dot{V} = \frac{E - V}{R} - i; \qquad L\dot{i} = V - \psi(i). \tag{24}$$

The equilibrium states are given by $\psi(i) = E - Ri$, and may be obtained as the intersections of the curve $V = \psi(i)$ with the line $V = E - Ri$ and there may be one or three intersection points. Let (i_0, V_0) be one of the points. Transfer the origin to the point, or equivalently substitute $i_0 + i$ and $V_0 + V$ for i, V in (24). At the

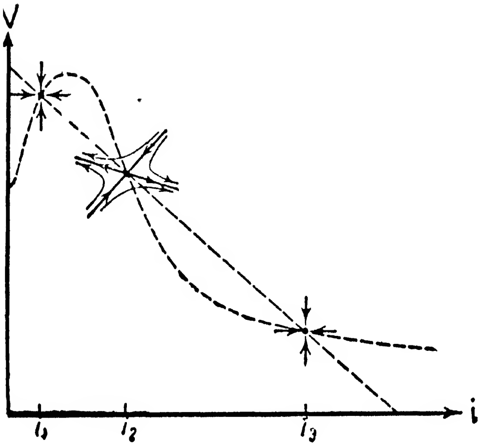


FIG. 255.

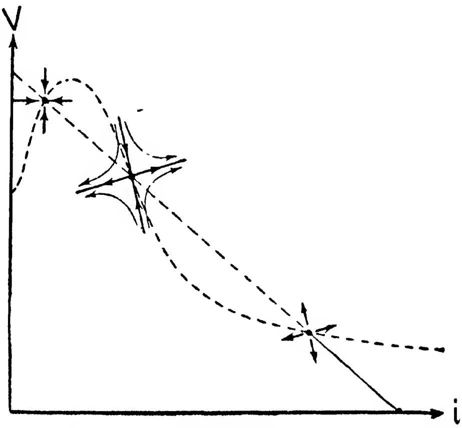


FIG. 256.

same time expand $\psi(i_0 + i)$ in power series keeping only the first two terms. As a consequence we have the first approximation system

$$\dot{V} = -\frac{V}{CR} - \frac{i}{C}, \qquad \dot{i} = \frac{V}{L} - \frac{\psi'(i_0)}{L}\,i. \tag{25}$$

The characteristic equation is, with $\psi' = \psi'(i_0)$,

$$\gamma^2 + \left(\frac{1}{RC} + \frac{\psi'}{L}\right)\gamma + \frac{1}{LC}\left(1 + \frac{\psi'}{R}\right) = 0 \tag{26}$$

Obviously, the character of the singular point, and consequently, the equilibrium state, depends on the sign of ψ', i.e. on the slope of the characteristic at the point corresponding to the given equilibrium state. Consider now the points 1,2,3 of the figure. Point 1 is always stable since $\psi'(i_1) > 0$ and hence all coefficients are positive. The roots of (26) are both real and negative or else both complex. Hence this equilibrium state is either a stable node or a stable focus according to the relationship between L, C, R and $\psi'(i_1)$. The diagram shows that $\psi'(i_2)$ is negative at the point 2 and in absolute value $> R$. Hence $1 + \psi'(i_2)/R < 0$ so that point 2 is a saddle-point and the corre-

sponding equilibrium state is always unstable. Finally, $\psi'(i_3) < 0$ at the point 3, but its absolute value $<R$. Consequently, $1 + \psi'(i_3)/R > 0$, i.e., the singular point 3 can also be either a focus or a node. This singular point is unstable when $|\psi'(i_3)/L| > 1/RC$, and stable when this inequality is reversed. Since $\psi'(i_3)$ is relatively small (the characteristic, although dropping, has a very slight slope), the equilibrium state 3 is always unstable when L is small, and always stable when C is small. In general, the passage from a stable to an unstable equilibrium state takes place at this point only for a certain "critical" value of the varying parameter (R, L or C). Thus, when there are three equilibrium states, their respective stability properties can be

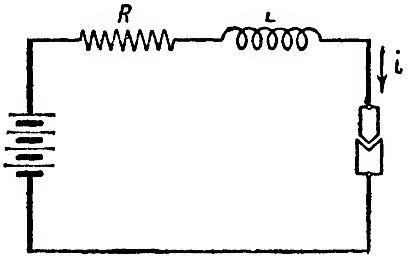


FIG. 257.

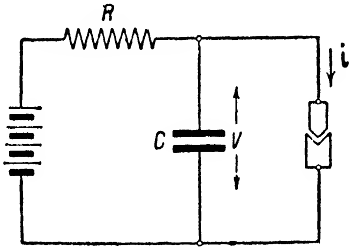


FIG. 258.

represented by one of the two possible combinations of Figs. 255 and 256. Let us examine now whether the character of these equilibrium states is affected by disregarding C or L. The corresponding diagrams are shown in Figs. 257 and 258. Both schemes were investigated in Chap. IV. Referring to it we find that the equilibrium states remain the same in all three cases: $L \neq 0$, and $C \neq 0$; $C = 0$ and $L \neq 0$; $C \neq 0$, $L = 0$. Nothing is changed when we pass from the first to the case where $C = 0$; as far as the stability of these equilibrium states is concerned, the equilibrium states 1 and 3 remain stable and the equilibrium state 2 remains unstable (as in the case when C is small but $\neq 0$). Passing however to the case $L = 0$, the equilibrium state 1 remains stable, and the equilibrium state 3 is unstable (as in the case when L is small but different from zero). The equilibrium state 2, however, is changed from stable for $L = 0$ to unstable for $L \neq 0$ (also $C \neq 0$), however small L may be. To see exactly what happens write the characteristic equation (26) as

$$(26') \qquad C \to 0: \quad C\gamma^2 + \gamma\left(\frac{1}{R} + C\frac{\psi'}{L}\right) + \frac{1}{LR}(\psi' + R) = 0,$$

$$(26'') \qquad L \to 0: \quad L\gamma^2 + \gamma\left(\frac{L}{CR} + \psi'\right) + \frac{1}{CR}(\psi' + R) = 0$$

where ψ' stands now for $\psi'(i_2)$. When C or L become zero, the order of the characteristic equation decreases and the new equation of first order has only one root. Since ψ' and $\psi' + R$ are both negative we see that $C = 0$ does not modify the stability situation at the point 2 (it remains unstable), but $L = 0$ does. In other words as regards stability it is safe to neglect the stray capacitance but not the stray inductance.

We see then that without an adequate verification, the equilibrium state under consideration could be easily mistaken for a stable state. This error has actually been made by Friedlander, who gave a voltaic arc in a circuit without inductance as an example of the existence of

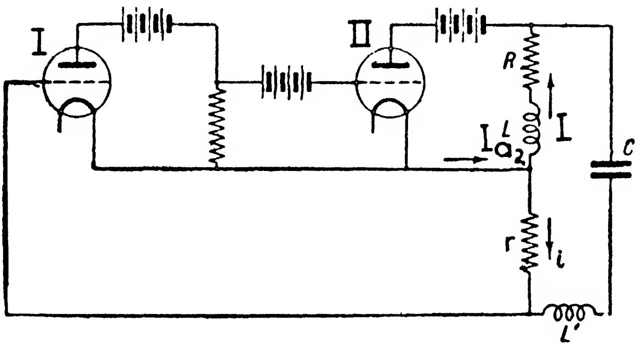


FIG. 259.

two stable equilibrium states on the phase curve. He considered the saddle-point as a stable equilibrium state while in reality it only "appears" to be stable. It is interesting to note that in another paper, Friedlander indicates the possibility of disrupting the stability of the equilibrium state by the introduction of an arbitrary small source of energy.

2. Consider now the circuit with electronic tubes represented in Fig. 259. We shall assume that the characteristic of the tube I has a long linear region. The function of this tube is to shift the phase of the applied voltage by 180° and to amplify this voltage by a factor k. We introduce a curvilinear characteristic for the tube II and assume that $I_{a_2} = \phi(kri)$. Upon neglecting the grid current and the plate reaction, one obtains for the circuit under investigation, two Kirchhoff equations:

$$I + i = I_{a_2}; \qquad L\dot{I} + RI - ri - L'\dot{i} - \frac{1}{C}\int i\,dt = 0$$

or after some simple transformations

$$(27) \qquad \dot{i} = \frac{R\phi(kri) - (R + r)i - V}{L + L' - Lkr\phi'(kri)}, \qquad \dot{V} = \frac{i}{C}.$$

The unique equilibrium state of this system is $i = 0$, $V_0 = R\phi(0)$. Let $\phi(x) = \phi(0) + x\phi'(0) + \cdots$. Setting $g = \phi'(0) =$ the grid-plate transconductance characteristic at the point $i = 0$ and replacing V by $V_0 + V$ in (27), we have the first approximation

$$\dot{i} = -\frac{\rho}{\lambda}i - \frac{V}{\lambda}, \qquad \dot{V} = \frac{i}{C},$$

$$\rho = r + R(1 - gkr), \qquad \lambda = L' + L(1 - gkr).$$

The characteristic equation is

$$\lambda\gamma^2 + \rho\gamma + \frac{1}{C} = 0$$

which is like (22) with b and c constant and, as we shall assume, $\rho \neq 0$. A slight change may occur however in that λ need not be positive. We have the following possibilities:

$\lambda > 0$. Node or focus which is stable when $\rho > 0$ and unstable otherwise.

$\lambda < 0$. Saddle point.

$\lambda = 0$. Stable point for $\rho > 0$, unstable point for $\rho < 0$.

Thus we see that for $\rho > 0$ and λ arbitrary but small the same contradiction arises as in the previous example. Here again neglecting the two stray inductances L,L' gives rise to the wrong stability decision. The error caused once more by "our being naive" is even less avoidable than in the previous example. In fact suppose first that there is a stray inductance in the resistance circuit, but not in the capacitance circuit, i.e. that $L \neq 0$, $L' = 0$. Suppose moreover that $1 - krg < 0$, but that R is so small that $\rho = r + R(1 - krg) > 0$. Then, according to our ordinary criteria, the equilibrium state is stable when $L = 0$, but for L very small it loses its stability and becomes a saddle-point. Therefore in a physical system such an equilibrium state is ruled out. However inductance, although small, must be present in the capacitor circuit, i.e. in a real system we necessarily have $L \neq 0$. If we take into account L', we do not disrupt the conditions of stability satisfied by the equilibrium state of the degenerate system when $L = 0$ and $\rho > 0$; we can even return the stability

to the equilibrium state which became unstable owing to the presence of the inductance L. In fact, if L' is large enough to have $\lambda = L' + L(1 - krg) > 0$, although $1 - krg < 0$, the equilibrium state which was unstable for $L \neq 0$ and $L' = 0$ will be stable. Thus stability is affected not only by the presence of parasitic parameters but also by their ratios.

Generally speaking there exists a range within which we do not know anything about the equilibrium state. In our example it is the range for ρ limited by the values: $r > \rho > 0$. In this range $1 - krg < 0$ and therefore, the stability of the equilibrium state depends on the values of parasitic parameters which cannot be taken into account. Therefore, when we are investigating the degenerate system and determine for this system the excitation condition, i.e. the condition of instability $\rho < 0$, we must take into account the regions of "undetermined" equilibrium states extending from $\rho = r$ to $\rho = 0$. It is quite possible that the so-called "parasitic self-excitation" which can appear and disappear in a given circuit without appreciably affecting the parameters of the circuit, is often due to small variations of the parameters in these regions of "undetermined" equilibrium states.

The presence of the parasitic parameters can also cause a "mix-up" regarding other stationary states. Our scheme may serve as an example of this situation. In the region where the equilibrium state is stable (when $\rho > 0$ and $\lambda > 0$) the system does not possess other stationary states. In other quadrants, where the equilibrium states are unstable, other stationary states can exist, namely, when $L' = 0$, there must necessarily exist in the second quadrant the stable "discontinuous limit-cycle," described in Chap. IV. However when the stray inductance L' is brought in, "jumps" become impossible and the "discontinuous limit-cycle" disappears (λ, however, may remain negative and the equilibrium state may remain unstable). In the third quadrant (when $\rho < 0$ and $\lambda < 0$), a continuous limit-cycle can never exist (because the unique singular point is a saddle point) when $L' = 0$ and $L \neq 0$. However if besides the parasitic L there exists also a stray L', we have $\lambda > 0$ when $\rho < 0$ (fourth quadrant) and the limit-cycle must necessarily exist (the singular point is an unstable focus or node and infinity is unstable). Thus, often we cannot assert definitely that there is a limit-cycle merely because we do not know the relationship between various parasitic parameters of the system.

CHAPTER VIII

Systems with Cylindrical Phase Surface

§1. Cylindrical Phase Space

The description of the behavior of a dynamical system by means of a phase space requires a one-one correspondence between the states of the system and the points of the space. This establishes a certain relationship between the nature of the physical system under consideration and the properties of the geometric form, representing the phase space of the system. Up to now we have been investigating physical systems (with one degree of freedom) whose phase space can be represented by a plane. In general, however, the plane will not be satisfactory, as is seen by considering the system consisting of the ordinary pendulum. Here the state of the system is determined by the angle of deviation from the equilibrium position, and the velocity of the pendulum. If we restrict ourselves to motions which do not exceed one revolution, no difficulties will arise if the phase space is taken as a plane. But when the angle of deviation is increased by a multiple of 2π we obtain a state identical with the original state. In a phase "plane" we would have an unlimited number of points corresponding to the same physical state of the system. Since the necessary one-oneness of the correspondence does not hold, a plane is not in general a suitable phase space for the pendulum.

In many cases the difficulties discussed above may be overcome by taking a cylinder as the phase space. In particular, for systems whose position is completely defined by their deviation angle, the condition of one-one correspondence is satisfied when the phase space is a cylinder whose axis coincides with the y-axis. Since such systems occur often, these cylindrical phase surfaces are of considerable interest. In the following paragraphs we discuss several systems of this sort and construct the corresponding phase portraits.

§2. A Conservative System

Consider an ordinary pendulum with linear friction (i.e. proportional to the velocity) under the action of a constant torque M_0. The motion of the pendulum is given by an equation of the form

$$I\ddot{\theta} + b\dot{\theta} + mga \sin \theta = M_0$$

where I is the moment of inertia, and b the moment of frictional forces when the angular velocity equals unity. The change of variable $\tau = t\sqrt{mga/I}$ gives

$$\frac{d^2\theta}{d\tau^2} + \alpha\frac{d\theta}{d\tau} + \sin\theta - \beta = 0 \tag{1}$$

where

$$\alpha = \frac{b}{\sqrt{Imga}} > 0 \quad \text{and} \quad \beta = \frac{M_0}{mga} > 0.$$

We will find later that equations such as (1) represent a large variety of physical systems.

In order to investigate (1) we introduce a new variable $z = d\theta/d\tau$, thus obtaining a system of two equations of the first order

$$\frac{dz}{d\tau} = -\alpha z - \sin\theta + \beta, \qquad \frac{d\theta}{d\tau} = z.$$

Eliminating τ we get one equation of the first order

$$z\frac{dz}{d\theta} = -\alpha z - \sin\theta + \beta \tag{2}$$

where θ and z are the coordinates of the cylindrical phase space.

In order to construct the phase portrait of the system under consideration we must examine the singular points, the separatrices, and the closed paths corresponding to the periodic motions. In the case of a cylindrical phase space there may occur two kinds of closed paths: (a) *Closed paths of the first kind.* They are the "ordinary" closed paths surrounding the equilibrium states without going around the cylinder, and they are completely analogous to the closed paths in the phase plane. (b) *Closed paths of the second kind.* They go around the cylinder without surrounding the equilibrium states. Obviously both kinds of closed paths correspond to periodic motions, and all of them must be known in order to construct the phase portrait on a cylinder.

The closed paths of the second kind correspond to periodic solutions of (2). Since the period is 2π, they must satisfy the condition $z(\theta_0 + 2\pi) = z(\theta_0)$ for all values of θ_0. In order to discover such periodic solutions, one might try to find two particular solutions $z_1(\theta)$ and $z_2(\theta)$ which for all values of θ_0 satisfy the inequalities

$$z_1(\theta_0 + 2\pi) \geqslant z_1(\theta_0), \qquad z_2(\theta_0 + 2\pi) \leqslant z_2(\theta_0).$$

Then, if there are no singular points between the paths corresponding to these two solutions, and since the solutions are a continuous function of the initial conditions, we see that between $z_1(\theta)$ and $z_2(\theta)$ there must exist a periodic solution for which

$$z(\theta_0 + 2\pi) = z(\theta_0).$$

Consider first the special case where $\alpha = 0$, so that the system is conservative. In this case the equation (2) takes the form

$$z\frac{dz}{d\theta} = \beta - \sin\theta,$$

which upon integration yields the solution

$$\tfrac{1}{2}z^2 = \cos\theta + \beta\theta + \frac{C}{2},$$

or

$$z = \pm\sqrt{2(\cos\theta + \beta\theta) + C}.$$

In order to facilitate the construction of the paths on the cylinder, we shall consider also an auxiliary (θ,y)-plane on which we trace the curve $y = 2(\cos\theta + \beta\theta)$. We then develop the cylindrical phase surface and place it under the auxiliary (θ,y)-plane. The construction of the paths in the plane (θ,z) is then reduced to the following: give to C different values, take the square root of $C + y$, and mark these values on the z-axis on both the positive and negative side. For a given value of C, there will correspond to every value of y for which $y + C > 0$, two points on the (θ,z)-plane, whereas for those y for which $y + C < 0$, z will be imaginary, and there will exist no corresponding points on the (θ,z)-plane.

The above construction will give different results for different values of β. When $\beta = 0$, i.e. $y = 2\cos\theta$, we obtain the picture represented in Fig. 260. For $C = -2$ we obtain one point $\theta = 0$, $z = 0$, which is a center. For $-2 < C < 2$ we have a series of concentric closed paths with a vertical tangent at $z = 0$. For $C = 2$ we get two curves which cross each other at the points $\theta = \pi$ and $\theta = -\pi$. These intersections are saddle points and the curves themselves separatrices. For $C > 2$ we have curves which never cross the θ axis, and finally for $C < -2$ there are no paths whatever. On the cylindrical phase surface the picture is as represented in Fig. 261. All the curves situated within the separatrices are closed curves surrounding

the center. They correspond to closed paths (periodic motions) of the first kind. The curves passing outside the separatrices ($C > 2$) are closed paths of the second kind.

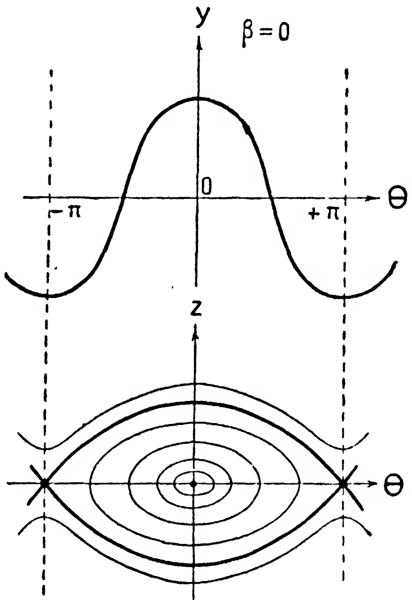


FIG. 260.

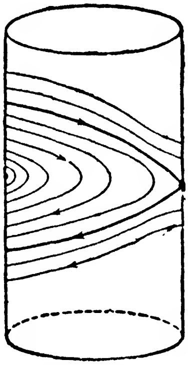


FIG. 261.

When $\beta \neq 0$ we obtain different pictures according as $\beta < 1$, $\beta = 1$, or $\beta > 1$. In any case we begin by constructing on the auxiliary plane the curve

$$y = 2 \cos \theta + 2\beta\theta. \tag{3}$$

Case (*a*): $\beta < 1$. Here (3) has a maximum at the points where $\theta = \arcsin \beta + 2k\pi$ and a minimum at the points where $\theta = -\arcsin \beta$

$+ (2k + 1)\pi$. Then we obtain (Fig. 262) a center, a saddle point, and one separatrix. On the cylinder the picture is as represented in Fig. 263. The curves within the separatrix are closed curves corre-

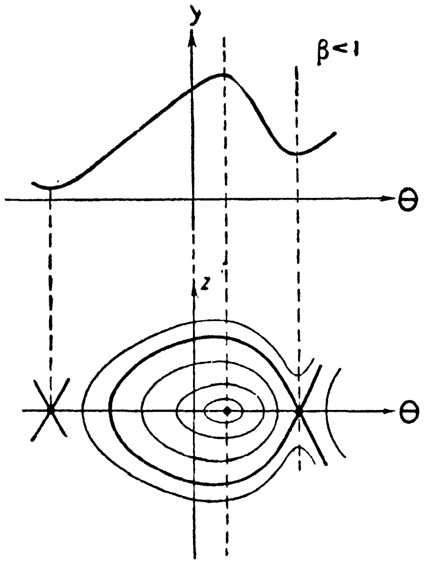


FIG. 262.

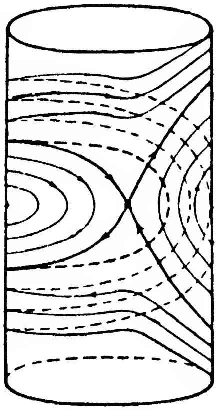


FIG. 263.

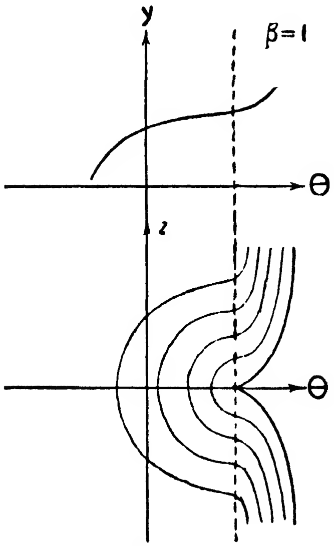


FIG. 264.

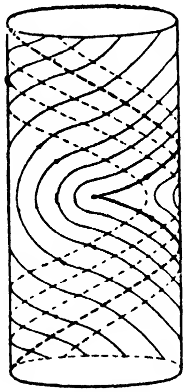


FIG. 265.

sponding to periodic motions. The curves situated outside the separatrix do not close on the cylinder since, when θ is increased by 2π, z does not return to its initial value and is, in fact, increased in

absolute value. Consequently, there are no closed paths of the second kind.

Case (*b*): $\beta = 1$. In this case (3) possesses neither a maximum nor a minimum, but has an inflection point with a horizontal tangent when $\theta = \pi/2$. Thus we obtain (see Fig. 264) one singular point of higher order. In this case there are no closed paths whatever (Fig. 265).

Case (*c*): $\beta > 1$. Here (3) increases monotonically and possesses neither extrema nor inflection points. Hence there are neither singular points (Fig. 266) nor closed paths on the cylinder (Fig. 267).

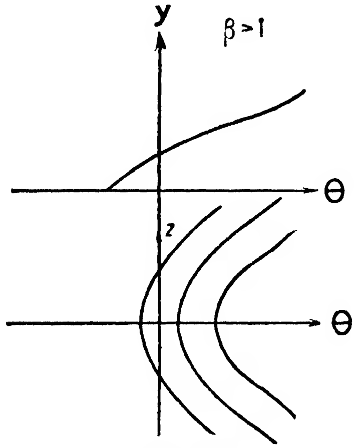


FIG. 266.

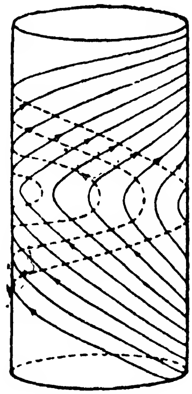


FIG. 267.

The physical interpretation of the above is clear. When $\beta < 1$, the constant moment is smaller than the largest moment of gravity and displaces the lower equilibrium state by less than $\pi/2$. Then, for sufficiently small initial deviations (and initial velocities), oscillations around this displaced equilibrium state are possible. If the initial deviation is large, then the action of the constant exterior moment will cause the pendulum to pass through the equilibrium position, to move farther in the direction of the constant moment, and the velocity of the pendulum will increase after every revolution.

On the other hand, when $\beta > 1$, the external moment is greater than the largest moment of gravity. In this case oscillations are altogether impossible, and regardless of the initial conditions the pendulum will finally rotate in the direction of the constant moment. Its velocity, although not varying monotonically, increases after every revolution.

§3. A Non-Conservative System

We now pass to the case $\alpha \neq 0$, so that the system is nonconservative. The differential equation of the paths on the cylinder may now be written

$$z\frac{dz}{d\theta} = -\alpha z - \sin\theta + \beta, \tag{4}$$

or also

$$\frac{dz}{d\theta} = \frac{-\alpha z - \sin\theta + \beta}{z}.$$

It cannot be integrated directly, and we are forced to a mere qualitative study of the paths.

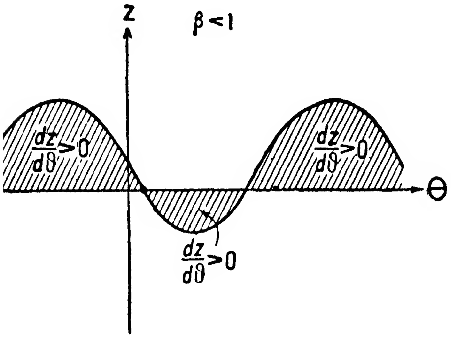


Fig. 268.

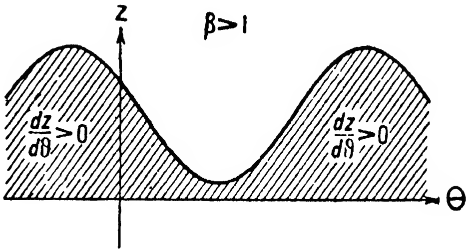


Fig. 269.

The isocline $dz/d\theta = 0$ has the equation

$$z = \frac{\beta - \sin\theta}{\alpha} \tag{5}$$

and represents a displaced sinusoid. It crosses the θ-axis only when $\beta < 1$ (Fig. 268). Also $dz/d\theta > 0$ between the sinusoid and the θ-axis (i.e. in the shaded regions of Figs. 268 and 269), and $dz/d\theta < 0$ elsewhere. The coordinates of the singular points are defined by the equations

$$\beta - \sin\theta = 0, \qquad z = 0,$$

and are as follows:

$$\begin{array}{lll} \text{Points } B_k: & \theta = 2k\pi + \theta_0, & z = 0 \\ \text{Points } A_k: & \theta = (2k-1)\pi - \theta_0, & z = 0 \end{array}$$

where $\theta_0 = \arcsin\beta$. We leave out of consideration the very special case $\beta = 1$, so that θ_0 (the least angle in absolute value whose sine is β) is such that $0 < \theta_0 < +\pi/2$.

Consider first the points B_k. Let $\theta = \theta_0 + 2k\pi + \eta$ and develop $\sin\theta$ in powers of η. If we keep only the term of first order in η, the behavior of the system in the neighborhood of B_k is given by the linear equation

$$\frac{dz}{d\eta} = \frac{-\alpha z - \eta\cos\theta_0 + \cdots}{z}.$$

The characteristic equation is

$$S^2 + \alpha S + \cos\theta_0 = 0$$

and its roots are

$$S_{1,2} = -\frac{\alpha}{2} \pm \sqrt{\frac{\alpha^2}{4} - \cos\theta_0}.$$

Consequently, the singular point is:

a focus when $\alpha^2 < 4\cos\theta_0$,
a node when $\alpha^2 > 4\cos\theta_0$,

and, as we have seen, when $\alpha = 0$ the singular point becomes a center.

Consider now the singular point A_k, so that $\theta = (2k - 1)\pi - \theta_0 + \eta$ and develop $\sin\theta$ in powers of η. Then as above,

$$\frac{dz}{d\eta} = \frac{-\alpha z + \eta\cos\theta_0 + \cdots}{z}.$$

The corresponding characteristic equation is

$$S^2 + \alpha S - \cos\theta_0 = 0,$$

and its roots are

$$S_{1,2} = -\frac{\alpha}{2} \pm \sqrt{\frac{\alpha^2}{4} + \cos\theta_0}. \tag{6}$$

Since $\cos\theta_0 > 0$ the roots are both real and of opposite sign, so that the singular point is a saddle point. The slopes of the separatrices at the saddle point are the solutions of

$$\mu^2 + \alpha\mu - \cos\theta_0 = 0$$

so that their values are

$$\mu = -\frac{1}{2}\alpha \pm \sqrt{\frac{\alpha^2}{4} + \cos\theta_0}. \tag{7}$$

Notice that, owing to $0 < \theta_0 < +\pi/2$, we have $\cos \theta_0 \neq 0$. Hence when $\alpha = 0$ the characteristic roots are still distinct and of opposite sign, and so the singular point under consideration remains a saddle point.

The next question to take up is the existence of periodic solutions. Here again we consider separately $\beta > 1$ and $\beta < 1$.

Case I: $\beta > 1$. As already mentioned, to prove the existence of periodic solutions it will suffice to find two particular solutions $z_1(\theta)$ and $z_2(\theta)$ of the basic differential equation (4) such that for some θ_1

$$z_1(\theta_1 + 2\pi) \leqslant z_1(\theta_1), \tag{A}$$
$$z_2(\theta_1 + 2\pi) \geqslant z_2(\theta_1). \tag{B}$$

Actually it is convenient to observe that it is sufficient to show that (A) holds for some θ_1, and (B) for some other, not necessarily the same, so that the values of θ_1 for which the one or the other holds is quite immaterial. For suppose that (A) holds for a given θ_1. Thus the function $\phi(\theta) = z_1(\theta + 2\pi) - z_1(\theta)$, which is continuous, is negative for $\theta = \theta_1$. Suppose that it could be positive for $\theta = \theta_2$. Then $\phi(\theta_3) = 0$ for some θ_3 between θ_1 and θ_2 or $z_1(\theta_3 + 2\pi) = z_1(\theta_3)$, from which follows that $z_1(\theta)$ is itself periodic since only one path passes through each point of the phase cylinder. Hence, if it is not, $z_1(\theta + 2\pi) - z_1(\theta)$ is always negative and so in (A) one may choose θ_1 arbitrarily. Similarly, of course, as regards (B). This property is used systematically in the sequel.

Consider now any solution $z_1(\theta)$ such that for some θ' we have

$$z_1(\theta') > (1 + \beta)/\alpha.$$

Since $|\sin \theta'| \leqslant 1$, and $\beta > 1$, the corresponding point is above the sinusoid (5), and so in the region where $dz/d\theta < 0$ (Fig. 270). The path must then be as in Fig. 270, rising constantly to the left of θ', and so we must have

$$z_1(\theta') \leqslant z_1(\theta' - 2\pi).$$

Thus (A) holds with $\theta_1 = \theta' + 2\pi$. To find a suitable $z_2(\theta)$ satisfying (B) consider the path through the point A (Fig. 271) having the coordinates $\theta = \pi/2$, $z = (\beta - 1)/\alpha$, (a point where $(\beta - \sin \theta)/\alpha$ has a minimum). We follow this path to the right of the point A. Since $dz/d\theta > 0$ between the sinusoid and the axis, the curve must go upwards as θ increases, and cross the sinusoid at a certain point Q. At this point the path has a horizontal tangent, since the sinusoid is the isocline $dz/d\theta = 0$. Then the path goes down and crosses the line

$\theta = 5\pi/2$ at a point M, which is never lower than the point B. This follows since the point at which the path next crosses the sinusoid is easily seen to be either B or beyond B. Thus the path under investigation corresponds to a solution for which $z_2(\pi/2 + 2\pi) \geqslant z_2(\pi/2)$, so that it satisfies condition (B) for $\theta_1 = \pi/2$.

Since in this case $\beta > 1$, there are no singular points and so there must exist between z_1 and z_2 a periodic solution for which $z_0(\theta + 2\pi) =$

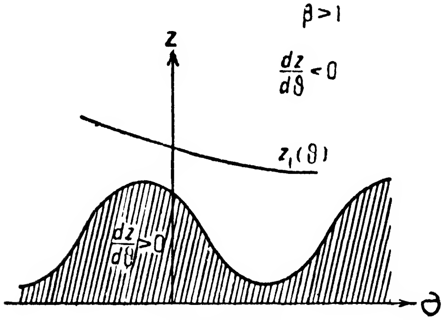


FIG. 270.

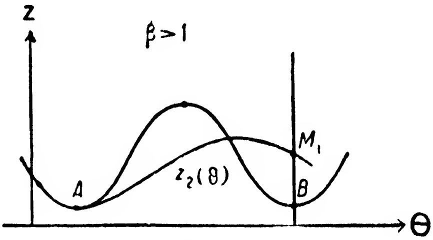


FIG. 271.

$z_0(\theta)$. We will now show that this periodic solution is unique. Integrating both sides of (4) from θ_1 to $\theta_1 + 2\pi$ we get

$$\tfrac{1}{2}[z(\theta_1 + 2\pi)]^2 - \tfrac{1}{2}[z(\theta_1)]^2 = -\alpha \int_{\theta_1}^{\theta_1+2\pi} z\, d\theta + 2\pi\beta.$$

Since for the periodic solution $z_0(\theta_1 + 2\pi) = z_0(\theta_1)$, this gives

$$\int_{\theta_1}^{\theta_1+2\pi} z_0(\theta)\, d\theta = \frac{2\pi\beta}{\alpha}. \tag{8}$$

Let us assume that there exist two periodic solutions z_{01} and z_{02}. Since they cannot intersect, we always either have $z_{01} > z_{02}$ or $z_{02} < z_{01}$. But by (8) this implies

$$\frac{2\pi\beta}{\alpha} = \int_{\theta_1}^{\theta_1+2\pi} z_{01}(\theta)\, d\theta > \int_{\theta_1}^{\theta_1+2\pi} z_{02}(\theta)\, d\theta = \frac{2\pi\beta}{\alpha}$$

(or with the inequality sign reversed) which is obviously impossible.

Thus we have a unique periodic solution. Since its path surrounds the cylinder it is of the second kind (Fig. 272). Closed paths (periodic motions) of the first kind are ruled out because they must surround singular points, and there are none here.

Case II: $\beta < 1$. In this case, as in the previous one, the existence of the solution z_1 satisfying $z_1(\theta_1 + 2\pi) \leqslant z_1(\theta_1)$ can be proved imme-

diately. In order to obtain a suitable $z_2(\theta)$ we shall investigate two paths (Fig. 273): the curve Γ_1 passing through the singular point A_1 with a positive inclination, and the curve Γ_2 passing through the point A_2 with a negative inclination. The slope of Γ_1 at A_1 is given by (7) as:

$$m_1 = -\frac{\alpha}{2} + \sqrt{\frac{\alpha^2}{4} + \cos\theta_0}$$

so that at this point the inclination of Γ_1 is smaller than that of the sinusoid.[1] Since the sinusoid is the isocline $dz/d\theta = 0$, Γ_1 crosses it

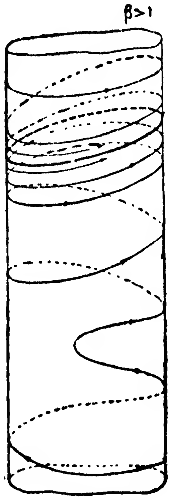


FIG. 272.

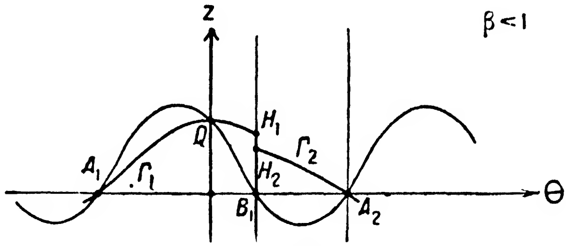


FIG. 273.

at the point Q and has a horizontal tangent there. The curve Γ_1 crosses the straight line $\theta = \theta_0$ at the point H_1. The curve Γ_2 leaves the point A_2 with a negative slope and crosses the straight line $\theta = \theta_0$ at the point H_2. Let the ordinates of these points be h_1 and h_2. When

[1] The inclination of the sinusoid at this point is equal to $\cos\theta_0/\alpha$. The magnitude m_1 is given by the equation

$$m_1^2 + m_1\alpha = \cos\theta_0.$$

If we replace m_1 by $\cos\theta_0/\alpha$, we obtain on the left side of the equation

$$\frac{\cos^2\theta_0}{\alpha^2} + \cos\theta_0 > \cos\theta_0.$$

Hence $m_1 < \dfrac{\cos\theta_0}{\alpha}$.

α is sufficiently small we have always $h_1 < h_2$, so that the path Γ_1, and those above but sufficiently close to Γ_1, satisfy the condition

$$z_2(\theta_1) \lessgtr z_2(\theta_1 + 2\pi).$$

Consequently, when α is sufficiently small, there will exist a periodic solution z_0 for which

$$z_0(\theta_1 + 2\pi) = z_0(\theta_1).$$

As in Case I, we can show that this solution is unique.

It can be shown that, when α is sufficiently large, we have $h_2 > h_1$, and that there exists a unique $\alpha = \alpha_0$ for which $h_1 = h_2$. Also, one

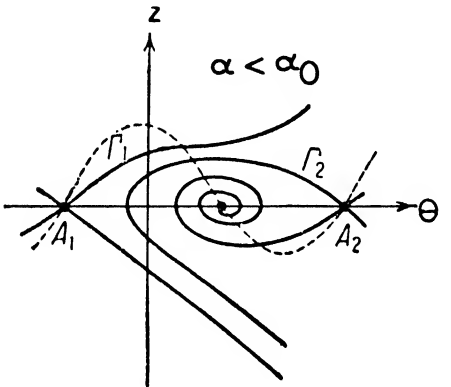


FIG. 274.

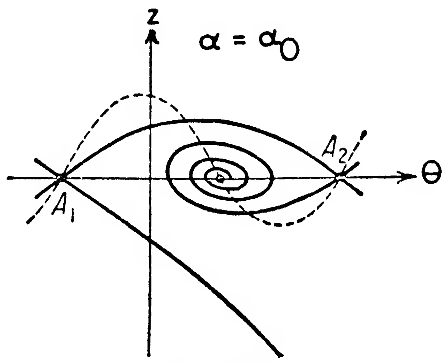


FIG. 275.

can show that periodic solutions cannot exist when $\alpha > \alpha_0$; and that, when $\alpha < \alpha_0$, the unique solution which exists is situated entirely in the region $z > 0$. Finally, it can be proved that in the case under investigation closed paths of the first kind are again ruled out. This is done by considering the form of the separatrices corresponding to $\alpha < \alpha_0$ (Fig. 274), $\alpha = \alpha_0$ (Fig. 275), $\alpha > \alpha_0$ (Fig. 276). Thus when $\beta < 1$ we obtain a unique periodic solution of the second kind when $\alpha < \alpha_0$ (Fig. 278); but no periodic solutions exist if $\alpha > \alpha_0$ (Fig. 277).

The periodic solutions of the second kind obtained (a) for all α, and $\beta > 1$, and (b) for $\alpha < \alpha_0$ and $\beta < 1$, are stable since all neighboring motions tend toward them. However, in case (a) the periodic solution becomes stable for all initial conditions, whereas in (b) there exists a region of initial conditions which lead the system to a state of rest (i.e. a stable focus). This region is shown in Fig. 279.

The above results may easily be interpreted physically. If the system experiences friction proportional to the velocity ($\alpha \neq 0$) and is acted upon by a constant moment of rotation, the work spent to

overcome the frictional forces increases together with the velocity, while the work of the external forces (other than gravity) remains constant. Therefore, if $\beta > 1$ so that the constant moment of the

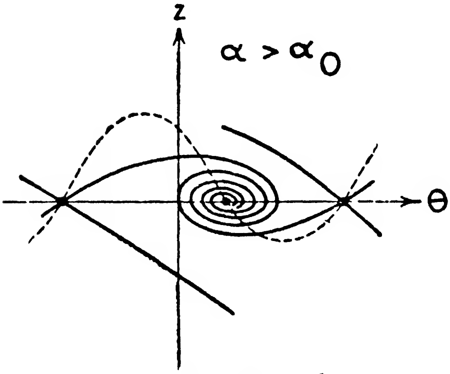


Fig. 276.

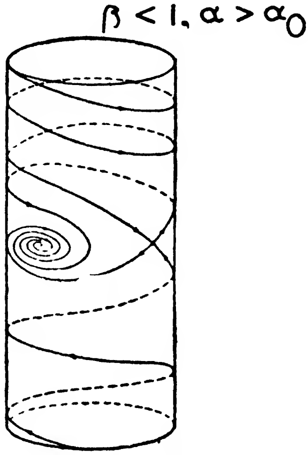


Fig. 277.

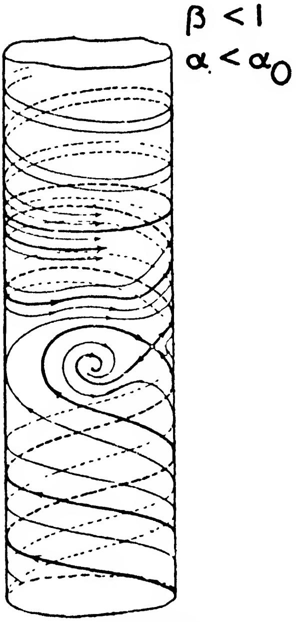


Fig. 278.

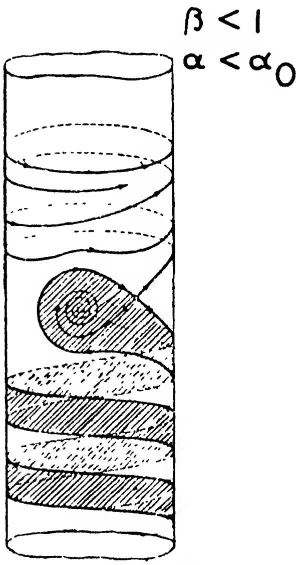


Fig. 279.

external forces exceeds the largest value of the moment of the forces of gravity, then whatever the initial conditions it will unwind the pendulum until an equilibrium is reached between the energy dissipated on friction and the work of the external forces. Conversely,

when $\beta < 1$, the external moment is so small as to be incapable of turning over the pendulum by itself. Thus the pendulum can start to move only under certain conditions. The motion can become periodic only under the condition that the energy dissipated on friction during one revolution will ultimately be equal to the work accomplished by the external forces creating the constant moment. Since at the same time we must have adequate initial conditions, in the form of a sufficiently high initial velocity, the loss of energy due to friction (for a given α) cannot be arbitrarily small. Thus, in order that the loss of energy on friction during one revolution will not exceed a certain value (equal to the work done by the exterior forces during one revolution), α must remain suitably small, i.e. smaller than a certain critical value α_0.

§4. OTHER SYSTEMS WHICH GIVE RISE TO A CYLINDRICAL PHASE SPACE

We will now discuss briefly two problems which give rise to a differential equation similar to (1) and wherein a similar analysis to the above may be applied. Consider first a system consisting of a synchronous motor connected in parallel with generators. Let θ be the angle between the axis of the magnetic field of the stator and the axis of the magnetic field of the rotor. Let M_0 represent the moment acting upon the motor as a result of load, which we here assume to be constant. Since this moment tends to slow down the rotor, we consider θ as being positive when the rotor is lagging with respect to the stator. Then if we assume that the resistive moment due to frictional forces and electric damping is proportional to the angular velocity, and that the moment resulting from the interaction of the rotor and stator fields is a function $f(\theta)$ of the angle θ, we obtain the equation

$$I\ddot{\theta} = M_0 - b\dot{\theta} - f(\theta). \tag{9}$$

It is to be noted that, since the moment $f(\theta)$ acts in such manner as to tend to decrease the angle θ, $f(\theta)$ has the same sign as θ, and $f(0) = 0$. This function $f(\theta)$ which characterizes the interaction of the rotor and stator fields is sinusoidal if certain simplifying assumptions are made. Then (9) will be essentially the same as (1).

Consider now a generator working in a circuit in parallel with other machines. Let θ denote the angle of advance of the rotor of the generator under investigation with respect to the rotors of the other motors, while M_0 represents the constant moment relative to the motor driving the generator. Here also M_0 tends to increase θ.

The damping moment is, as before, equal to $-b\dot{\theta}$. Furthermore, in the case of a generator working in the general circuit, the displacement of θ creates an electro-mechanical moment acting upon the given generator as a result of the functioning of other motors in the same circuit.[1] This moment is a function of θ tending to decrease the value of θ. Therefore it must be equal to $-f(\theta)$ where $f(\theta)$ has the same sign as θ. Again, under certain assumptions it may be considered as behaving like sin θ, and we are led once more to an equation similar to (1).

[1] It is to be noted that, when the system is immobile, the equation of motion of the rotor has the following form

$$I\ddot{\psi} = f(\omega t - \psi) - b\dot{\psi},$$

where ω is the angular velocity of the axis of the magnetic field of the stator and ψ the rotation angle of the rotor. Setting $\theta = \omega t - \psi$ yields the equation in the text.

CHAPTER IX

Quantitative Investigation of Non-Linear Systems

§1. The van der Pol Method of Approximation

We now pass to the quantitative investigation of non-linear systems. In its present state the theory applies satisfactorily only to two types of systems, which, however, are of considerable practical interest. The first type includes approximately conservative systems, among which the approximately sinusoidal systems are most interesting. The second type includes systems with discontinuous oscillations and has previously been discussed quantitatively. In this chapter we consider only the simplest kind of systems of the first type, viz., systems which are approximately linear and conservative.

For such systems the equation of motion can be written[1]

$$\ddot{x} + x = \mu f(x,\dot{x}),$$

[1] The equation of a system similar to a harmonic oscillator, when expressed in ordinary variables, has the form:

(a) $$\frac{d^2v}{d\tau^2} + \omega_0^2 v = \mu F\left(v, \frac{dv}{d\tau}, \mu\right)$$

where τ is the time, ω_0 the angular frequency, v a dependent variable (voltage or current, for example), and μ the so-called small parameter, which we shall assume to be dimensionless and which defines the closeness of the system under investigation to a linear conservative system. If we introduce the dimensionless independent variable $t = \omega_0\tau$ and the dimensionless dependent variable $x = v/v_0$ (where v_0 is a fixed quantity of the same dimensionality as v, saturation voltage or saturation current, for example), (a) becomes:

$$\ddot{x} + x = \mu \frac{1}{v_0\omega_0^2} F(v_0x, v_0\omega_0\dot{x};\mu)$$

or, writing

$$\frac{1}{v_0\omega_0^2} F(v_0x, v_0\omega_0\dot{x};\mu) = f(x,\dot{x};\mu),$$

we have

(b) $$\ddot{x} + x = \mu f(x,\dot{x};\mu).$$

Note that for simplicity the theory is developed here for the particular case when $f(x,\dot{x};\mu)$ is independent of μ. The formulas developed for the first approximation of solutions of (1) are, however, valid for the more general case when $f(x,\dot{x};\mu)$ is a polynomial in x, $\dot{x}$ and μ, provided that we replace $f(x,y)$ by $f(x,y,0)$.

or equivalently

(1) $$\dot{x} = y, \qquad \dot{y} = -x + \mu f(x,y).$$

Here μ is a dimensionless positive parameter, which we shall assume to be small. The size of μ determines how close the system is to a linear conservative one. We shall also assume that $f(x,y)$ is a polynomial in x and y. We examine (1) by what we call the van der Pol method. It consists in replacing (1) by simpler "auxiliary" equations—which we call van der Pol equations—whose solutions approxi-

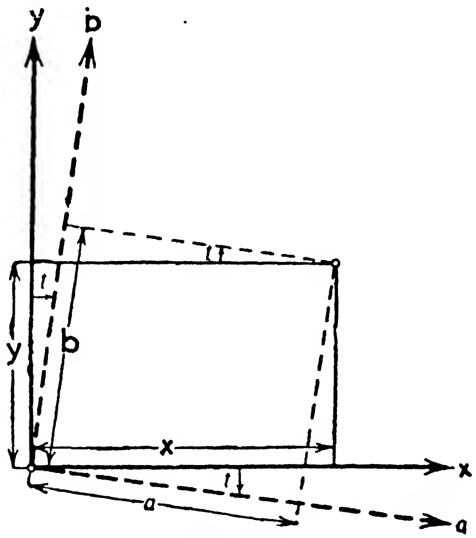


Fig. 280.

mate the solutions of (1) over arbitrary time intervals, when μ is small. The method takes account of the non-linearity of (1); in fact, the auxiliary equations are also non-linear.

To find the auxiliary, or van der Pol equations, consider a coordinate system (a,b) in the phase plane (Fig. 280), rotating in the clockwise direction with angular velocity unity. For $\mu = 0$ (1) becomes the harmonic oscillator whose paths are circles with center at the origin. The representative point moves with the rotating a,b-axes and every point of the a,b-plane is an equilibrium state in this case. The transformation between the rotating a,b-axes and the fixed x,y-axes (see Fig. 280) is[1]

[1] In order to pass to the coordinates a,b we need not have recourse to the rotating axes. In fact, let us look for a solution of (1) of the form:

(a) $$x = a(t) \cos t + b(t) \sin t.$$

Since $x(t)$ does not determine completely $a(t)$ and $b(t)$ by (a), we can impose

$$x = a \cos t + b \sin t$$
$$y = -a \sin t + b \cos t.$$

Using these relations to change variables in (1) we obtain

$$\dot{x} = y = -a \sin t + b \cos t + \dot{a} \cos t + \dot{b} \sin t$$

and therefore

$$\dot{a} \cos t + \dot{b} \sin t = 0.$$

Differentiating again we have

$$\ddot{x} + x = \mu f(x,\dot{x}) = -\dot{a} \sin t + \dot{b} \cos t.$$

Solving the last two equations for the derivatives of a and b we find after some simplifications

$$\dot{a} = -\mu f(a \cos t + b \sin t, -a \sin t + b \cos t) \sin t,$$
$$\dot{b} = \mu f(a \cos t + b \sin t, -a \sin t + b \cos t) \cos t.$$

Expanding the right sides into Fourier series gives

$$(2) \quad \begin{cases} \dot{a} = \mu \left(\dfrac{\phi_0(a,b)}{2} + \phi_1(a,b) \cos t + \bar{\phi}_1(a,b) \sin t \right. \\ \qquad\qquad \left. + \phi_2(a,b) \cos 2t + \bar{\phi}_2(a,b) \sin 2t + \cdots \right), \\ \dot{b} = \mu \left(\dfrac{\psi_0(a,b)}{2} + \psi_1(a,b) \cos t + \bar{\psi}_1(a,b) \sin t \right. \\ \qquad\qquad \left. + \psi_2(a,b) \cos 2t + \bar{\psi}_2(a,b) \sin 2t + \cdots \right), \end{cases}$$

where ϕ_i, $\bar{\phi}_i$, ψ_i, $\bar{\psi}_i$ are the corresponding Fourier coefficients of the functions:

$$-f(a \cos t + b \sin t, -a \sin t + b \cos t) \sin t,$$
$$+f(a \cos t + b \sin t, -a \sin t + b \cos t) \cos t.$$

Since the transformation from (x,y) to (a,b) involves t, the new system of equations may not be autonomous although the original one was.

Let us investigate with (2) the auxiliary system[1]

---

another condition, say

$$\text{(b)} \qquad \dot{a} \cos t + \dot{b} \sin t = 0.$$

Clearly (a) and (b) are equivalent to the transformation to a rotating system.

[1] The system (3) in a and b is obtained rather simply by van der Pol. He sets

$$\dot{a} = \mu \frac{\phi_0(a,b)}{2}, \qquad \dot{b} = \mu \frac{\psi_0(a,b)}{2}, \tag{3}$$

obtained by neglecting the "oscillatory" terms. The system (3) has the great advantage that when we pass to polar coordinates the variables can be separated. Its justification will be found in Appendix B. Since

$$\frac{\phi_0(a,b)}{2} = -\frac{1}{2\pi}\int_0^{2\pi} f(a\cos\xi + b\sin\xi, -a\sin\xi + b\cos\xi)\sin\xi\,d\xi,$$

$$\frac{\psi_0(a,b)}{2} = \frac{1}{2\pi}\int_0^{2\pi} f(a\cos\xi + b\sin\xi, -a\sin\xi + b\cos\xi)\cos\xi\,d\xi,$$

if we set $a = K\cos\theta$, $b = K\sin\theta$, so that

$$K = +\sqrt{a^2+b^2}, \qquad \tan\theta = \frac{b}{a},$$

we have[1]

$$\text{(4a)}\quad \dot{K} = \mu\Phi(K), \qquad \text{(4b)}\quad \dot{\theta} = \mu\Psi(K),$$

---

as in the text $x = a\cos t + b\sin t$. Then the equation becomes

$$\underline{\ddot{a}\cos t} + 2\dot{b}\cos t + \underline{\ddot{b}\sin t} - 2\dot{a}\sin t$$
$$= \mu f(a\cos t + b\sin t, -a\sin t + b\cos t + \underline{\dot{a}\cos t} + \underline{\dot{b}\sin t}).$$

He neglects the underlined terms because a and b are slowly varying functions of time. Then he expands f in a Fourier series, assuming a and b constant and equates the coefficients of $\cos t$ and of $\sin t$. This gives (3).

[1] Multiplying the formulas for ϕ_0 and ψ_0 by a and b respectively, and adding, we obtain

$$\frac{1}{2}\frac{d}{dt}K^2 = \frac{\mu}{2\pi}\int_0^{2\pi} f(a\cos\xi + b\sin\xi, -a\sin\xi + b\cos\xi)(-a\sin\xi + b\cos\xi)\,d\xi.$$

Letting $a\cos\xi + b\sin\xi = K\cos(\xi+\delta)$, we have further:

$$\frac{1}{2}\frac{d}{dt}K^2 = \frac{-\mu}{2\pi}\int_0^{2\pi} f(K\cos(\xi+\delta), -K\sin(\xi+\delta))K\sin(\xi+\delta)\,d\xi.$$

Hence, since the integrand is periodic, assuming $\xi + \delta = u$ and dividing by K the equation (4a) follows. Multiplying the formulae for ϕ_0 and ψ_0 by $-b$ and a respectively, and adding, we obtain

$$a\dot{b} - b\dot{a} = K^2\dot{\theta} = \frac{\mu}{2\pi}\int_0^{2\pi} f(a\cos\xi + b\sin\xi, -a\sin\xi + b\cos\xi)(a\cos\xi + b\sin\xi)\,d\xi.$$

In similar fashion this gives equation (4b).

where

$$\Phi(K) = -\frac{1}{2\pi}\int_0^{2\pi} f(K\cos u, -K\sin u)\sin u\,du,$$

$$\Psi(K) = \frac{1}{2\pi K}\int_0^{2\pi} f(K\cos u, -K\sin u)\cos u\,du.$$

It is easy to investigate the auxiliary system. Consider (4a). The qualitative behavior for such an equation is, as we have seen, completely defined by the situation and the character of the equilibrium states on the corresponding phase line. The coordinates of these equilibrium states are the roots of the equation

$$\Phi(K) = 0 \tag{5}$$

or

$$\int_0^{2\pi} f(K\cos u, -K\sin u)\sin u\,du = 0. \tag{6}$$

The equilibrium state $K = K_i$ is stable if $\Phi'(K_i) < 0$ or if[1]

$$\int_0^{2\pi} f_y(K_i\cos u, -K_i\sin u)\,du < 0 \tag{7}$$

and is unstable if $\Phi'(K_i) > 0$. All the other motions are, as we know, either asymptotic to an equilibrium state for both $t \to \pm\infty$, or asymptotic to an equilibrium state for $t \to +\infty$ and tend to infinity for $t \to -\infty$, or vice versa (see p. 144).

For the approximating motions (4) we can always find analytic expressions. In fact, from (4a) follows:

[1] In fact:

$$\begin{aligned}\Phi'(K) &= -\frac{1}{2\pi}\int_0^{2\pi} f_x\cos u\sin u\,du + \frac{1}{2\pi}\int_0^{2\pi} f_y\sin^2 u\,du\\ &= \frac{-1}{2\pi K}\int_0^{2\pi}(f_x K\sin u + f_y K\cos u)\cos u\,du + \frac{1}{2\pi}\int_0^{2\pi} f_y\,du\\ &= \frac{1}{2\pi K}\int_0^{2\pi}\frac{d}{du}(f\cos u)\,du + \frac{1}{2\pi K}\int_0^{2\pi} f\sin u\,du + \frac{1}{2\pi}\int_0^{2\pi} f_y\,du\\ &= \frac{1}{2\pi K}\int_0^{2\pi} f\sin u\,du + \frac{1}{2\pi}\int_0^{2\pi} f_y\,du\end{aligned}$$

and, since $\Phi(K_i) = 0$,

$$\Phi'(K_i) = \frac{1}{2\pi}\int_0^{2\pi} f_y(K_i\cos u, -K_i\sin u)\,du.$$

$$\mu(t - t_0) = \int_{K_0}^{K} \frac{dK}{\Phi(K)}$$

where $K = K_0$ for $t = t_0$. Hence, solving for K, we have:

$$K = K\{\mu(t - t_0)\}.$$

Now consider equation (4b) and the representation of the motion in the a,b-plane. We must distinguish two cases. In the first case, often met in practice.

$$\Psi(K) \equiv 0. \tag{8}$$

Here (4b) yields $\theta = \text{constant} = \theta_0$, so the paths are straight lines through the origin with inclination $\theta = \text{constant}$. The motion is the

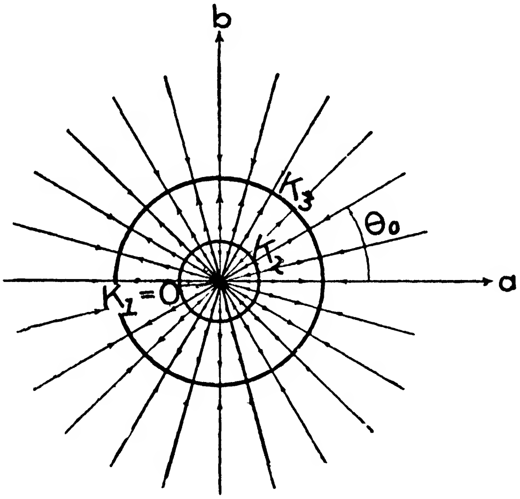


FIG. 281.

same along each line and is defined by (4a). The roots of equation (5), $K = K_i$, give the radii of the circles which represent the curves of equilibrium states. The picture of three equilibrium states corresponding to equation (4a) is shown in Fig. 281.

If we pass from our rotating coordinates (a,b) to the original coordinates (x,y), it is easy to see that the circles of equilibrium states in the a,b-plane become circular limit-cycles having the same radii K_i.

The motion of the representative point along a cycle of radius K_i is given by

$$\begin{aligned} x &= a \cos t + b \sin t = K_i \cos \theta_0 \cos t + K_i \sin \theta_0 \sin t \\ &\qquad = K_i \cos (t - \theta_0), \\ y &= -a \sin t + b \cos t = -K_i \cos \theta_0 \sin t + K_i \sin \theta_0 \cos t \\ &\qquad = -K_i \sin (t - \theta_0), \end{aligned}$$

where θ_0 is arbitrary. The fact that the phase angle θ_0 of the periodic motions along a given limit-cycle is arbitrary corresponds in the (a,b)-plane to the fact that the equilibrium states of the auxiliary equations form whole circumferences.

Clearly the limit-cycle is orbitally stable if the corresponding equilibrium states are stable, and conversely. The other paths, which are linear segments in the a,b-plane, are transformed in the x,y-plane into spirals which approach the limit-cycles either for $t \to +\infty$ or $t \to -\infty$.

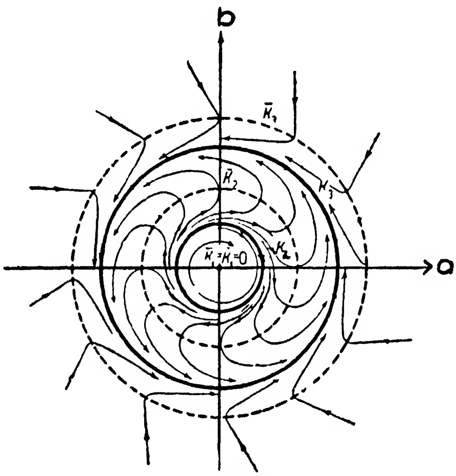


FIG. 282.

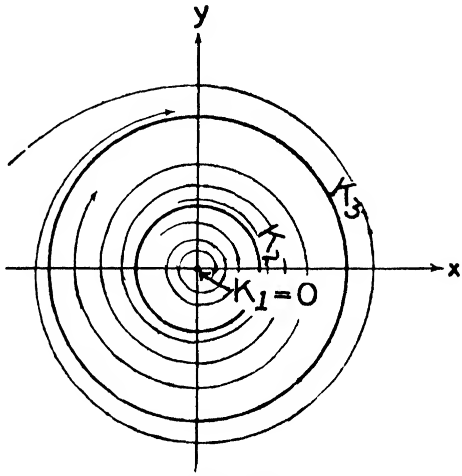


FIG. 283.

Now consider the second case, $\Psi(K) \not\equiv 0$. We assume that the equation $\Psi(K) = 0$ has several roots, $\bar{K}_1,\bar{K}_2, \cdots ,\bar{K}_m$, all different from $K_1,K_2, \cdots ,K_n$. Then from (4) it is easy to see that the equilibrium states of (4a) in the a,b-plane correspond to circular limit cycles with radii $K_1,K_2, \cdots ,K_n$. The motion of the representative point in the a,b-plane along a limit cycle of radius K_j is given by

$$\begin{aligned} a &= K_j \cos \{\mu\Psi(K_j)t + \theta_0\} \\ b &= K_j \sin \{\mu\Psi(K_j)t + \theta_0\}. \end{aligned}$$

The stability or instability of the limit-cycle is determined by the stability or instability of the corresponding equilibrium state given by equation (4a) and the direction of rotation is defined by the sign of $\Psi(K_j)$.

All other curves are spirals tending to the limit-cycles either for $t \to +\infty$ or $t \to -\infty$ (Fig. 282). If in this second case we use stationary coordinates, we obtain a picture similar to that in the first

case. There are again a series of limit-cycles of radii $K_1, K_2, \cdots, K_n$. The motion along the limit-cycles $K = K_j$, is given by:

$$(9) \quad \begin{cases} x = a \cos t + b \sin t = K_j \cos \{[1 - \mu\Psi(K_j)]t - \theta_0\}, \\ y = -a \sin t + b \cos t = -K_j \sin \{[1 - \mu\Psi(K_j)]t - \theta_0\}. \end{cases}$$

This differs from the first case only in having a definite correction for the frequency $\Delta\omega = -\mu\Psi(K_j)$ which, in the first approximation with respect to μ, corresponds to the correction for the period $\tau = 2\pi\mu\Psi(K_j)$. The other trajectories are again spirals which wind around the limit-cycles.

We now propose to pass from information about (3) to information about (1). It can be shown that for small μ the solutions of (3) are close to the solutions of (1). More precisely, if $\Phi(K)$ has simple roots, then (1) has limit-cycles close to the circles of radii K_j, and no others. (This will be shown in §5, in connection with the Poincaré method.) These limit-cycles correspond to periodic solutions, stable in the sense of Liapounoff if $\Phi'(K_j) < 0$. If $\Psi(K_j) = 0$ the correction for the period (from that of the harmonic oscillator) begins with terms proportional to μ^2, but if $\Psi(K_j) \neq 0$ the correction begins with $2\pi\Psi(K_j)\mu$.

Concerning (4) one should bear in mind that $\Phi(K)$ is the constant term in the Fourier expansion of $f(K \cos u, -K \sin u) \sin u$, and $\Psi(K)$ the constant term in the expansion of $f(K \cos u, -K \sin u) \cos u$. Assuming that $f(x,y)$ is a polynomial, these constant terms can be found with the help of standard trigonometric formulas.

It is often more convenient to replace (4a) by

$$\dot{\rho} = \bar{\Phi}(\rho),$$

where

$$\rho = K^2, \qquad \bar{\Phi}(\rho) = 2\sqrt{\rho}\,\Phi(\sqrt{\rho}).$$

§2. Applications of the van der Pol Method

1. The vacuum tube under soft working conditions. Consider a vacuum tube oscillator with a tuned grid circuit (Fig. 284). The Kirchhoff equation determining the current in the oscillating circuit of the tube gives

$$(10) \qquad L\dot{i} + Ri + \frac{1}{C}\int i\,dt = M\dot{I}_a.$$

We assume that the plate current depends on the grid voltage alone (i.e. we neglect the plate reaction) and shall consider first a cubic characteristic for the tube, i.e. a polynomial in the "dimensionless

voltage" v:

$$I_a = V_s(\alpha_1 v + \beta_1 v^2 - \gamma_1 v^3). \tag{11}$$

Here $v = V_g/V_s$, V_g and V_s are the grid and saturation voltages respectively, and $\alpha_1,\beta_1,\gamma_1$ are coefficients whose common dimension is that of conductance. Since $V_g = \frac{1}{C}\int i\,dt$ and $v = \frac{1}{CV_s}\int i\,dt$, we can write (10) in the form

$$LC\ddot{v} + RC\dot{v} + v = \frac{M}{V_s}\dot{I}_a.$$

Then using (11), making elementary transformations, and writing t for $\omega_0 t$, we obtain

$$\ddot{v} + v = (\alpha + 2\beta v - 3\gamma v^2)\dot{v}. \tag{12}$$

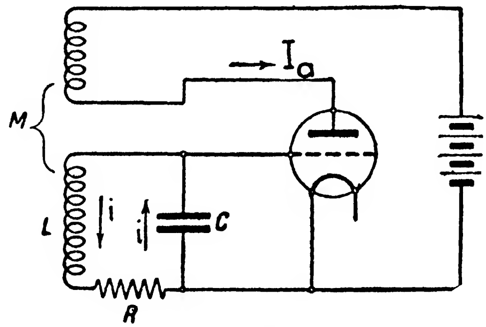


FIG. 284.

Here

$$\omega_0^2 = \frac{1}{LC}, \qquad \alpha = (M\alpha_1 - RC)\omega_0, \qquad \beta = M\beta_1\omega_0, \qquad \gamma = M\gamma_1\omega_0,$$

and α,β,γ are dimensionless parameters which we shall assume to be small and of the same order. Then, setting

$$\alpha = \mu\alpha', \qquad \beta = \mu\beta', \qquad \gamma = \mu\gamma',$$

where μ represents the small parameter of the previous discussion ($\mu \geqslant 0$) and α',β',γ' are dimensionless quantities close to unity. We shall assume μ small enough for the solutions of the van der Pol approximation to have the essential properties of the solutions of (12), particularly with respect to periodic solutions and their stability. This, of course, imposes limitations on α,β,γ, and hence on the class of problems discussed here.

Let $v = x$, $\dot{v} = y$, and consider the system of two linear equations corresponding to (12),

$$\dot{x} = y; \qquad \dot{y} = -x + \mu(\alpha' + 2\beta' x - 3\gamma' x^2)y. \tag{13}$$

From (3) and (4) the auxiliary equations are

$$\dot{a} = \mu \frac{a}{2}\left(\alpha' - \frac{3\gamma'}{4}(a^2 + b^2)\right), \qquad \dot{b} = \mu \frac{b}{2}\left(\alpha' - \frac{3\gamma'}{4}(a^2 + b^2)\right)$$

or in polar coordinates K,θ:

$$\dot{K} = \mu \frac{K}{2}\left(\alpha' - \frac{3\gamma'}{4}K^2\right), \qquad \dot{\theta} = 0. \tag{14}$$

The radii of the limit-cycles in the x,y-plane are given by

$$\Phi(K) = \frac{K}{2}\left(\alpha' - \frac{3\gamma'}{4}K^2\right) = 0.$$

We assume $\gamma' > 0$ and shall consider separately the cases $\alpha' > 0$ and $\alpha' < 0$.

If $\alpha' < 0$, the only significant root of Φ is $K = 0$. Then the limit-cycle degenerates to a point, which is a stable singular point since $\Phi'(0) = \alpha'/2 < 0$. It is easy to see that the paths are spirals tending to the origin as $t \to +\infty$. Thus we have in the x,y-plane a picture characteristic of damped oscillations.

If $\alpha' > 0$, Φ has two significant roots,

$$K_1 = 0, \qquad K_2 = +\sqrt{\frac{4\alpha'}{3\gamma'}} = +\sqrt{\frac{4\alpha}{3\gamma}}.$$

Since $\Phi'(0) = \alpha'/2 > 0$, $\Phi'(K_2) = -\alpha' < 0$, K_1 corresponds to an unstable singular point while K_2 corresponds to a stable limit-cycle.

All the paths other than the limit-cycle can be divided into two classes: the paths spiralling towards the limit-cycle from the outside as $t \to +\infty$ and going to infinity as $t \to -\infty$, and the paths spiralling towards the limit-cycle from the inside as $t \to +\infty$ and towards the singular point when $t \to -\infty$. The resulting picture is characteristic for a self-oscillating system under soft working conditions.

If α' has a positive value which is decreased continuously (for example, by increasing R), γ' remaining constant, the square of the radius of the limit-cycle $K_2^2 = 4\alpha/3\gamma$ will decrease according to a linear law (Fig. 285). When $\alpha = 0$, the limit-cycle disappears and

merges with the unstable focus, which then becomes stable. In this case zero is a branch point of the parameter α. If α varies continuously from negative to positive values, oscillations begin at $\alpha = 0$, their amplitude increasing continuously. If α decreases, the amplitude of

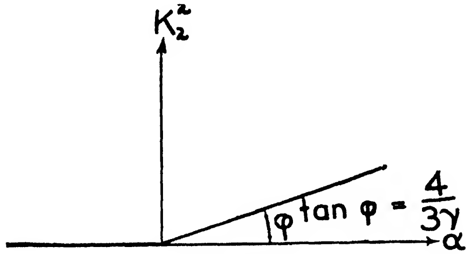


FIG. 285.

the oscillations decreases, reaching zero when $\alpha = 0$. The system then behaves like a damped oscillator (Fig. 286). This is called soft generation of oscillations in contrast to the case (hard generation) when the oscillations appear suddenly and have a finite amplitude (though the parameter itself varies continuously). The auxiliary equations will enable us to obtain approximate expressions for the

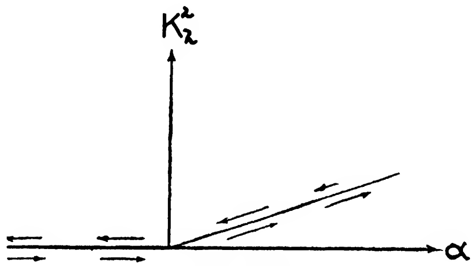


FIG. 286.

oscillations. Integrating (14), we find (assuming $\alpha > 0$)

$$K = K_0\sqrt{\frac{C_1e^{\alpha t}}{C_1e^{\alpha t} + K_0^2}}, \qquad \left(K_0^2 = \frac{4\alpha}{3\gamma}\right),$$

$$\theta = \text{constant} = \theta_0.$$

Hence

$$x = K_0\sqrt{\frac{e^{\alpha(t-\theta_0)}}{e^{\alpha(t-\theta_0)} + C}}\cos(t - \theta_0)$$

$$y = -K_0\sqrt{\frac{e^{\alpha(t-\theta_0)}}{e^{\alpha(t-\theta_0)} + C}}\sin(t - \theta_0)$$

where $C = \dfrac{K_0^2}{C_1e^{\alpha\theta_0}}$.

We call this an approximate expression for the general integral of (13) because it has two arbitrary constants, C and θ_0 ($C = 0$ corresponds to the limit-cycle, $C = \infty$ to the equilibrium state). Note that (11) contained a square term which does not appear in the general solution of the zero approximation. (The effect of this term appears only in the next approximation.) This is a general property of all even terms of the characteristic. It is due to the fact that the development of the sines and cosines of an even power contains only the sines and cosines of the even multiple angles and hence does not contain the fundamental (resonance) frequency.

2. The vacuum tube under hard working conditions. Consider again the vacuum tube described by (10) but assume now a fifth-order characteristic

$$I_a = V_s(\alpha_1 v + \gamma_1 v^3 - \delta_1 v^5). \tag{15}$$

Setting

$$\alpha = (M\alpha_1 - RC)\omega_0, \qquad \gamma = M\gamma_1\omega_0, \qquad \delta = M\delta_1\omega_0,$$

then also

$$\alpha = \mu\alpha', \qquad \gamma = \mu\gamma', \qquad \delta = \mu\delta',$$

we have this time, instead of (12), the relation

$$\ddot{v} + v = \mu(\alpha' + 3\gamma' v^2 - 5\delta' v^4)\dot{v}. \tag{16}$$

Then by (3) and (4) the auxiliary system is:

$$\begin{cases} \dot{a} = \mu \dfrac{a}{2}\left(\alpha' + \dfrac{3}{4}\gamma'(a^2 + b^2) - \dfrac{5}{8}\delta'(a^2 + b^2)^2\right), \\ \dot{b} = \mu \dfrac{b}{2}\left(\alpha' + \dfrac{3}{4}\gamma'(a^2 + b^2) - \dfrac{5}{8}\delta'(a^2 + b^2)^2\right), \end{cases}$$

or in polar coordinates K,θ:

$$\begin{cases} \dot{K} = \mu \dfrac{K}{2}\left(\alpha' + \dfrac{3}{4}\gamma' K^2 - \dfrac{5}{8}\delta' K^4\right), \\ \dot{\theta} = 0. \end{cases} \tag{17}$$

The radii of the limit-cycles (of the zero approximation) are the roots of

$$\Phi(K) = \frac{K}{2}\left(\alpha' + \frac{3}{4}\gamma' K^2 - \frac{5}{8}\delta' K^4\right) = 0.$$

If we assume $\gamma' > 0$, then $\delta' > 0$, while α' can be positive or negative.

Since 0 is a root of Φ, the origin is an equilibrium state, and, since $\Phi'(0) = \alpha'/2$, it is stable when $\alpha' < 0$ and unstable when $\alpha' > 0$. The radii of the limit-cycles which do not degenerate to points are the roots of

$$\alpha + \tfrac{3}{4}\gamma K^2 - \tfrac{5}{8}\delta K^4 = 0.$$

In general this gives two values, K_1^2 and K_2^2 (we only care for the positive roots), which yield the radii, K_1 and K_2, of the limit-cycles. Constructing a graph (Fig. 287) with α on the horizontal axis and K^2 (square of the radii of the limit-cycles) on the vertical axis, this

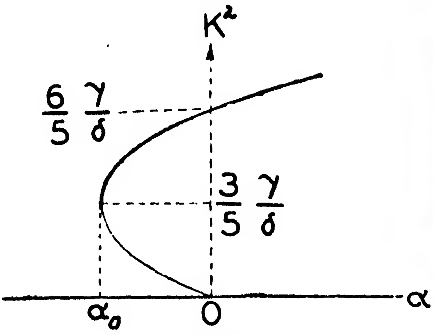


FIG. 287.

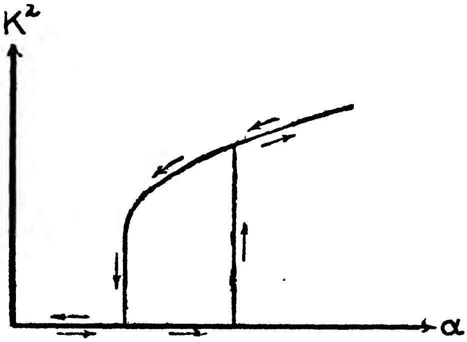


FIG. 288.

equation (for constant γ and δ) determines a parabola. The curve crosses the K^2-axis at $K^2 = 0$ and $K^2 = 6\gamma/5\delta$; its vertex is on the line $r = 3\gamma/5\delta$ and it opens to the right. The stability of the limit-cycles with $\Phi'(K) < 0$ shows that the upper half of the parabola corresponds to stable limit-cycles, the lower half to unstable limit-cycles. In Fig. 287 the stable portions are marked in heavy lines; this figure indicates the nature of the phase portrait for different α, with γ and δ constant.

If at the beginning the tube is not excited, for example, when α has a negative value (Fig. 287), and α is increasing, the oscillation of the final amplitude becomes excited when α reaches zero. A further increase in α leads to a monotonic increase of amplitude. When α decreases, the amplitude diminishes until α reaches the abscissa α_0 of the vertex. When $\alpha = \alpha_0$ the oscillations stop. Contrary to the case of soft generation, the appearance and disappearance of oscillations takes place for different values of the excitation coefficient and we can observe a certain "lengthening" (a type of hysteresis during the transition from the equilibrium state to the periodic motion). Furthermore, oscillations with different (but finite) amplitudes appear and disappear. In general, we obtain a typical picture of the hard

generation of oscillations (Fig. 288). If initially the system is at the left of the vertex, the oscillations can never be excited without changing the parameters. If, however, the system is between α_0 and $\alpha = 0$, then, although self-excitation is not present in the system, it can experience oscillations. To excite the oscillations it is necessary however, to give the system a shock. As we shall see later, oscillations can be generated in the system only if the square of the initial deviation from equilibrium is larger than the unstable branch of the parabola. When α is positive, self-excitation of oscillations takes place; oscillations can be created for an arbitrarily small initial deviation from equilibrium.

Note that (17) can be integrated in the same way as for a tube under soft working conditions and we can obtain a solution characterizing quantitatively the generation of oscillations.

§3. The Method of Poincaré (Method of Perturbations)

The method in question is fundamental in the discovery of periodic solutions in systems depending upon a parameter μ. It is supposed that for a certain value $\mu = \mu_0$ the system has a periodic solution. Poincaré has given a general method for finding the periodic solutions when μ is near μ_0. Replacing, if need be, μ by $\mu - \mu_0$, one may suppose $\mu_0 = 0$, and so one is looking for the possible periodic solutions when μ is small. It is true that the method applies only to analytical systems, but this will be more than sufficient for our purpose. We shall investigate the application of the method to the nearly linear equation

$$\ddot{x} + x = \mu f(x,\dot{x}), \tag{18}$$

or to the equivalent system

$$\dot{x} = y, \qquad \dot{y} = \mu f(x,y) - x. \tag{19}$$

For simplicity we shall also suppose that $f(x,y)$ is a polynomial. This is again ample for practical purposes. In point of fact, except for having to mention here and there "convergence," we could as well suppose that $f(x,y)$ is a power series in x and y.

Consider the reduced system

$$\ddot{x} + x = 0$$

or equivalently

$$\dot{x} = y, \qquad \dot{y} = -x. \tag{20}$$

The general solution of (20) is

$$x = K \cos (t - t_0), \qquad y = -K \sin (t - t_0) \tag{21}$$

and the paths are the circles

$$\Gamma_K: \quad x^2 + y^2 = K^2.$$

The particular solution (21) of the reduced system represents the curve cutting the x-axis at the point $(K,0)$ at time $t = t_0$. Now the solution of (19) cutting the x-axis at the point $(K + \beta,0)$ at time $t = t_0$ may be written

$$\begin{aligned} x &= K \cos (t - t_0) + \xi(K,\beta,\mu,t - t_0), \\ y &= -K \sin (t - t_0) + \eta(K,\beta,\mu,t - t_0), \\ \eta &= \dot{\xi}, \end{aligned}$$

and this is the general solution of (19). Counting the time from the crossing of the x-axis at $(K + \beta,0)$, or, what is the same, writing t for $t - t_0$, our solution is represented by

$$\begin{cases} x = K \cos t + \xi(K,\beta,\mu,t), \\ y = -K \sin t + \eta(K,\beta,\mu,t), \\ \eta = \dot{\xi}. \end{cases} \tag{22}$$

It is to be recalled that for $t = 0$ we have $x = K + \beta$, $y = 0$ and hence

$$\xi(K,\beta,\mu,0) = \beta, \qquad \eta(K,\beta,\mu,0) = 0, \tag{23}$$

and of course, since for $\beta = \mu = 0$ the solution reduces to (21) for $t_0 = 0$, we must have identically

$$\xi(K,0,0,t) = 0, \qquad \eta(K,0,0,t) = 0. \tag{24}$$

Since the system (19) is analytical, the solutions are analytical in μ,β, and may be expanded in power series for small values of the variables. It follows that ξ,η have the same property. By (24) they vanish identically for $\beta = \mu = 0$, and so the series are of the form

$$\begin{cases} \xi = A(t)\beta + B(t)\mu + C(t)\beta\mu + D(t)\mu^2 + \cdots, \\ \eta = \dot{\xi} = \dot{A}\beta + \dot{B}\mu + \dot{C}\beta\mu + \dot{D}\mu^2 + \cdots. \end{cases} \tag{25}$$

By Taylor's expansion for polynomials

$$\begin{aligned} f(K \cos t + \xi, -K \sin t + \eta) &= f(K \cos t, -K \sin t) \\ &+ \left\{ \xi f_x + \eta f_y + \tfrac{1}{2}(\xi^2 f_{x^2} + 2\xi\eta f_{xy} + \eta^2 f_{y^2}) + \cdots \right\}_{\substack{x = K \cos t \\ y = -K \sin t}}. \end{aligned} \tag{26}$$

We now substitute the solution (22) in the system (19) where ξ,η are the series (25). By means of (26) we obtain series in β,μ which must be identical. This yields the following relations between the coefficients of the series:

$$(27)\qquad \begin{cases} \ddot{A} + A = 0, & \ddot{C} + C = f_x A + f_y \dot{A}, \\ \ddot{B} + B = f(x,y), & \ddot{D} + D = f_x B + f_y \dot{B}, \\ x = K \cos t, & y = -K \sin t. \end{cases}$$

According to (23) for $t = 0$, ξ and η must reduce respectively to β and zero. Hence identically

$$(28)\quad A(0) = 1, \qquad \dot{A}(0) = B(0) = \dot{B}(0) = C(0) = \dot{C}(0) = D(0) \\ \dot{D}(0) = \cdots = 0.$$

The differential equations (27) together with the initial conditions (28) make it possible to determine exactly $A, \cdots, D, \cdots$. Each of these variables satisfies a differential equation of the form

$$\ddot{z} + z = \phi(t)$$

whose solution $z(t)$ such that $z(0) = \dot{z}(0) = 0$ is (by direct verification)

$$(29)\qquad z = \int_0^t \phi(u) \sin (t - u)\, du.$$

As it is repeatedly needed, we write down the derivative

$$(30)\qquad \dot{z}(t) = \int_0^t \phi(u) \cos (t - u)\, du.$$

The solution $A(t)$ with initial conditions $A(0) = 1$, $\dot{A}(0) = 0$ is immediately found to be $A(t) = \cos t$. The other coefficients are found by means of (29) to be

$$(31)\qquad \begin{cases} B(t) = \int_0^t f(x,y) \sin (t - u)\, du, \\ C(t) = \int_0^t \{f_x \cos u - f_y \sin u\} \sin (t - u)\, du, \\ D(t) = \int_0^t \{f_x B(u) + f_y B'(u)\} \sin (t - u)\, du, \\ \quad x = K \cos u, \qquad y = -K \sin u. \end{cases}$$

So far we have only prepared the ground for the periodic solutions. We return to them, and it is here that we find the crux and also the most interesting part of Poincaré's argument. We are looking for a periodic solution of (19) very near, when β and μ are small, to the solution $K \cos t, -K \sin t$ of (20), and so with a period very close to 2π. Let this period be written $2\pi + \tau(\beta,\mu)$, where τ is the "correction" to

the basic harmonic period 2π. We will expect then that $\tau(\beta,0) = 0$, whatever β, since for $\mu = 0$ we merely have the fixed harmonic period 2π. Now by taking β and μ small enough the solution (22) of (19) will hold in an arbitrarily large interval $0 \leqslant t \leqslant T$. Take $T = 4\pi$, and assume $|\tau| < \pi$. Thus the solution will hold for β,μ small enough in $0 \leqslant t \leqslant 2\pi + \tau$. It will be periodic in the manner stated when and only when

$$(32) \qquad x(2\pi + \tau) - x(0) = 0, \qquad y(2\pi + \tau) - y(0) = 0,$$

and these relations are the basic relations of Poincaré.

Let us substitute x,y from (22) in (32) with ξ,η given by (25), and upon so doing replace $\cos\tau$ and $\sin\tau$ by their series expansions. We then find, after some simplifications,

$$(33) \quad P(\tau) = \frac{-K\tau^2}{2} + B(2\pi)\mu + \dot{B}(2\pi)\tau\mu + C(2\pi)\beta\mu + D(2\pi)\mu^2 + \cdots = 0,$$

$$(34) \quad Q(\tau) = -K\tau + \dot{B}(2\pi)\mu + \ddot{B}(2\pi)\tau\mu + \dot{C}(2\pi)\beta\mu + \ddot{A}(2\pi)\tau\beta + \dot{D}(2\pi)\mu^2 + \cdots = 0.$$

These are then two equations to determine τ and β as functions of μ. Keeping only the terms of at most the first order in $Q(\tau)$ (the first two terms), we find for τ the approximation

$$(35) \qquad \tau = \frac{\dot{B}(2\pi)\mu}{K} = \frac{\mu}{K}\int_0^{2\pi} f(K\cos u, -K\sin u)\cos u\,du = 2\pi\mu\Psi(K),$$

where Ψ is as in (4). We assume here, of course, $K \neq 0$. In (33) there is only one term of order one (the second). Rejecting the rest we find

$$-B(2\pi) = \int_0^{2\pi} f(K\cos u, -K\sin u)\sin u\,du = 0,$$

or, finally, with Φ as in (4)

$$(36) \qquad \Phi(K) = 0.$$

This equation determines the radii of the generating circles. In other words, the circles Γ_K of the harmonic system must be specially selected, with radii satisfying (36), if they are to be "circles of departure" for periodic solutions of the basic system (19).

If $\dot{B}(2\pi) = 0$, then τ is at least of order two in μ. Setting $\tau = \kappa\mu^2$, we obtain

$$\begin{aligned} C(2\pi)\beta + D(2\pi)\mu &= 0, \\ -K\kappa\mu + \dot{C}(2\pi)\beta + \dot{D}(2\pi)\mu &= 0. \end{aligned}$$

As we are only interested in true generating circles, we may assume $K \neq 0$. The equations just written yield then for $C(2\pi) \neq 0$:

$$\beta = \frac{-D(2\pi)}{C(2\pi)}\mu, \qquad \kappa = \frac{\dot{D}(2\pi)C(2\pi) - D(2\pi)\dot{C}(2\pi)}{KC(2\pi)}.$$

Finally, if we introduce the values of A,C,β into (22) and return to the more general time origin t_0 (i.e., replace t back by $t - t_0$) we find up to terms in μ^2 the approximate expression for the periodic solution

$$(37)\quad \begin{cases} x(t) = \left(K - \dfrac{D(2\pi)}{C(2\pi)}\mu\right)\cos(t - t_0) \\ \qquad\qquad + \mu \displaystyle\int_0^t f(K\cos u, -K\sin u)\sin(t - t_0 - u)\,du. \\ y(t) = \dot{x}(t), \qquad \Phi(K) = 0. \end{cases}$$

This form of the solution suffers from the evident disadvantage that it has the period $2\pi + \tau$ only "up to terms in μ^2," but this is the price to pay for the approximation.

Let us now discuss, following Poincaré, the stability of the periodic solution corresponding to a definite root K_1 of $\Phi(K) = 0$. Let $x(t),y(t)$ represent a solution very near the periodic solution which we assume to correspond to a K very near K_1. Since successive crossings of the x-axis by the closed path corresponding to the periodic solution occur at times $0,2\pi + \tau$, where τ is small, those for the curve corresponding to $x(t),y(t)$ will occur successively at times $0,2\pi + \theta$, where θ is small and time is counted from the first crossing. We will have

$$\begin{aligned} Q(\theta) &= y(2\pi + \theta) - y(0) = 0, \\ P(\theta) &= x(2\pi + \theta) - x(0). \end{aligned}$$

Now in $P(\theta)$ the term of lowest order is $B(2\pi)\mu = 2\pi\mu\Phi(K)$. Assuming $\mu > 0$, as will happen most of the time (if $\mu < 0$ the stability conclusions must be reversed), we will have stability if $P(\theta)$ decreases as K increases (we come nearer the periodic solution when we start beyond it or below it), i.e. if $\Phi(K)$ passes through 0 decreasingly at K_1, or again if $\Phi'(K_1) < 0$. As we have already found earlier (§1), this stability condition takes the form

$$(38)\qquad \int_0^{2\pi} f_y(K_1\cos u, -K_1\sin u)\,du < 0.$$

However it was derived there by means of unrigorous approximations, whereas now, by Poincaré's method, the reasoning is essentially rigorous, and furthermore, in concept, it may be applied to many other situations as well.

In the calculations it is often convenient to bear in mind, as observed already in §1, that the integral in (38) is, up to a factor 2π, the constant term in the Fourier expansion of the function $f_y(K_1 \cos u, -K_1 \sin u)$. It is frequently easy to calculate this term directly.

A final remark concerning Poincaré's procedure. We have supposed throughout that the function f is analytic. However, if it fails to be so, it may be approximated with arbitrarily high accuracy by a polynomial f^*. As this does not affect the results very much, we may reason as before with f^* in place of f. However, in calculating, for example, $\Phi(K)$, the substitution of f^* for f does not appreciably affect Φ, and so we may actually apply the results (with some care) with f as it stands. This will be done in particular with the broken line characteristic in the second example to follow.

§4. APPLICATION OF THE POINCARÉ METHOD

1. The vacuum tube under soft working conditions. The investigation of the equation of an ordinary vacuum tube oscillation with a tuned grid circuit (Fig. 284) under soft working conditions illustrates the Poincaré method. We can limit the investigation of this case to a cubic characteristic for the tube and use (12) directly. We omit the square term of the characteristic because it is unimportant for approximately sinusoidal oscillations. Then for the "dimensionless voltage" we have

$$\ddot{v} + v = \mu(\alpha' - 3\gamma' v^2)\dot{v} = \mu f(v,\dot{v}), \tag{39}$$

$$f(x,y) = (\alpha' - 3\gamma' x^2)y. \tag{40}$$

Here the dots represent differentiation with respect to "dimensionless" time; the other symbols have the same meaning as in §3. In particular, $\alpha = \mu\alpha'$, $\gamma = \mu\gamma'$.

The form of (39) is suitable for the Poincaré method, so we apply it. The radius K of the generating circle is defined by

$$B(2\pi) = -\int_0^{2\pi} \left(-\alpha' K \sin u - \gamma' \frac{d}{du}(K^3 \cos^3 u)\right) \sin u \, du = 0. \tag{41}$$

This gives (to within a constant positive factor)

$$\Phi(K) = \left(\alpha' K - \frac{3\gamma' K^3}{4}\right) = 0.$$

Hence

$$K^2 = \frac{4\alpha'}{3\gamma'} = \frac{4\alpha}{3\gamma}.$$

We have here

$$\int_0^{2\pi} \left[-\alpha' K \sin u - \gamma' \frac{d}{du} (K^3 \cos^3 u) \right] \cos u \, du = 0$$

and so $\dot{B}(2\pi) = 0$. Hence the first approximation for τ is zero. We also find from their expressions

$$C(2\pi) = \pi \left[\alpha' - \frac{9\gamma' K^2}{4} \right] = -2\pi\alpha'; \qquad \dot{C}(2\pi) = 0,$$

$$\dot{D}(2\pi) = \frac{3\pi\alpha'\gamma' K^3}{32} = \pi\alpha'^2 K/8,$$

and hence finally

$$\tau = \pi\mu^2\alpha'^2/8 = \pi\alpha^2/8.$$

As in §3 stability is governed by

$$\Phi'(K) = \alpha' - \tfrac{9}{4}\gamma' K^2 < 0,$$

or finally by

$$K^2 > \frac{4}{9}\frac{\alpha'}{\gamma'} = \frac{4}{9}\frac{\alpha}{\gamma}.$$

We have seen that approximately $K^2 = \frac{4}{3}\frac{\alpha}{\gamma}$. Hence the periodic solution generated by the circle of radius $K = \sqrt{\frac{4}{3}\frac{\alpha}{\gamma}}$ is stable.

§5. The Vacuum Tube with Broken Line Characteristic

Let us apply the small parameter method to a vacuum tube whose characteristic is made up of two segments, one horizontal and one inclined. We shall see that in the case of such characteristics, or better for characteristics limited only on one side, stable oscillations may arise under certain conditions.

Assume that the characteristic of the tube has the form shown in Fig. 289 or 290 (so b may be positive or negative). In the case of a tuned grid the equations relative to the ("dimensionless") voltage

on the capacitor are of the form (see (11) and (12))

$$(42)\qquad \begin{cases} v < b: & \ddot{v} + v = -R\sqrt{\dfrac{C}{L}}\,\dot{v}, \\ v > b: & \ddot{v} + v = -R\sqrt{\dfrac{C}{L}}\,\dot{v} + \dfrac{MS}{\sqrt{LC}}\,\dot{v} = \dfrac{MS - RC}{\sqrt{LC}}\,\dot{v}. \end{cases}$$

We shall show that the conditions of self-excitation are:

$$\begin{aligned} &b > 0\text{: no self-excitation,} \\ &b < 0\text{: } \begin{cases} \text{no self-excitation if } RC > MS \\ \text{self-excitation if } RC < MS. \end{cases} \end{aligned}$$

The solution has the form

$$v = K \sin t.$$

We shall assume K positive; since the phase is arbitrary, this does not restrict the generality. The amplitude K is given by the condition

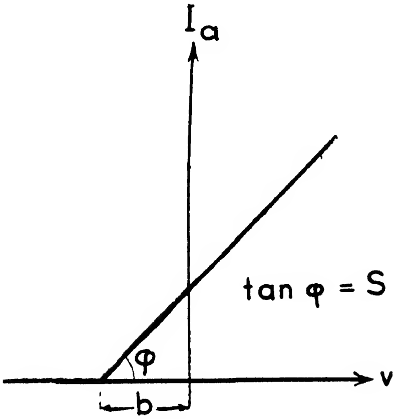


Fig. 289.

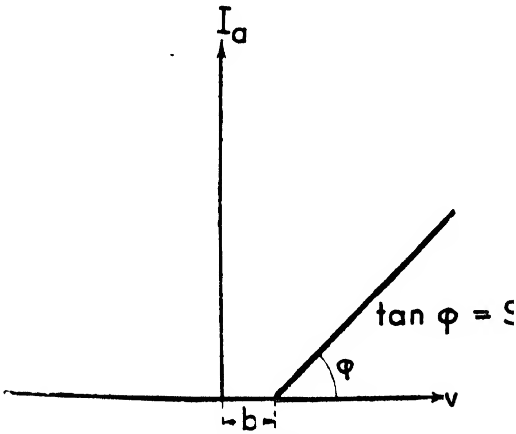


Fig. 290.

$B(2\pi) = 0$. To determine $B(2\pi)$ we must calculate according to (42) the values of $f(K \cos u, -K \sin u)$ corresponding to $b > 0$ and $b < 0$. We obtain

$$(43)\qquad B(2\pi) = -\pi R\sqrt{\frac{C}{L}} + \frac{2MS}{\sqrt{LC}} \int_{\xi}^{\frac{\pi}{2}} \cos^2 t\, dt = 0$$

where

$$(44)\qquad \xi = \arcsin \frac{b}{K}.$$

We are interested only in the values $-\pi/2 < \xi < +\pi/2$, and since K is positive, only in the values of ξ of the same sign as b. There-

fore for $b > 0$, ξ is between 0 and $\pi/2$, and for $b < 0$ between $-\pi/2$ and 0. Although the amplitude does not enter obviously in (43), it is not arbitrary but is defined by (44) as

$$K = b/\sin \xi$$

After integrating, (43) yields

$$\sin 2\xi = -2\pi \frac{RC}{MS} + \pi - 2\xi. \tag{45}$$

To solve this, consider (Fig. 291) the intersection of the curves

$$y = \sin 2\xi, \qquad y = -2\pi \frac{RC}{MS} + \pi - 2\xi.$$

The maximum negative slope of the sine curve is -2 while the second curve is a line of slope -2, displaced from the origin by $\pi/2 - \pi RC/MS$

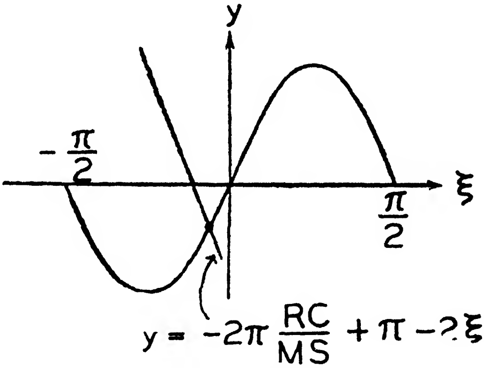


FIG. 291.

units along the ξ-axis. If $b > 0$, then

$$0 < \frac{\pi}{2} - \pi \frac{RC}{MS} < \frac{\pi}{2}, \tag{46}$$

and, if $b < 0$,

$$-\frac{\pi}{2} < \frac{\pi}{2} - \pi \frac{RC}{MS} < 0. \tag{47}$$

If these inequalities are not satisfied, periodic solutions do not exist. We have (46) only if $RC < MS/2$, and (47) only if $MS/2 < RC < MS$. Thus, if $RC < MS/2$ and $b > 0$, or if $MS/2 < RC < MS$ and $b < 0$, there is one limit-cycle. (These cases are illustrated in Figs. 292 and 293.)

If $RC > MS/2$ and $b > 0$, or if either $RC < MS/2$ or $RC > MS$ and $b < 0$, there are no limit-cycles. We note for future reference

that, if ξ_0, the root of (45), is positive ($b > 0$), then

(48) $$\xi_0 < \frac{\pi}{2} - \pi\frac{RC}{MS},$$

while, if ξ_0 is negative ($b < 0$), then

(49) $$\xi_0 > \frac{\pi}{2} - \pi\frac{RC}{MS}.$$

This follows from the negative slope of the line above.

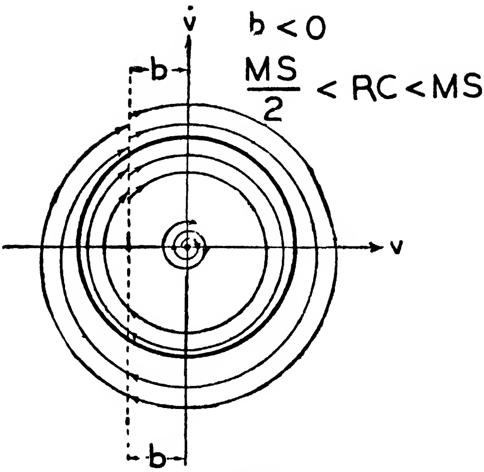


FIG. 292.

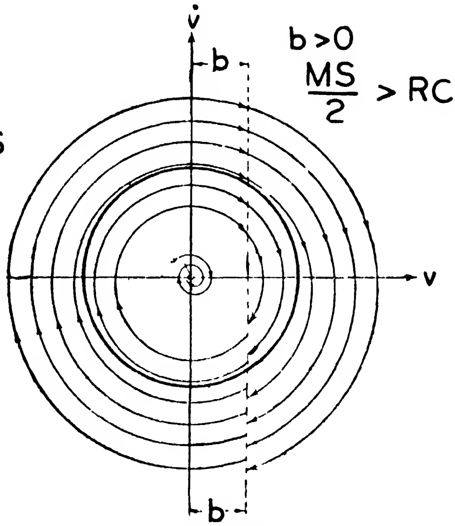


FIG. 293.

Consider now the stability of the periodic motions. The condition for stability is

$$-\pi R\sqrt{\frac{C}{L}} + \frac{MS}{\sqrt{LC}}\int_{\xi_0}^{\frac{\pi}{2}} dt < 0.$$

This leads to

$$-RC + \frac{MS}{2} - \frac{MS\xi_0}{\pi} < 0$$

or

$$\xi_0 > \frac{\pi}{2} - \pi\frac{RC}{MS}.$$

From (48) and (49) this holds for $b < 0$ and fails for $b > 0$. Hence the limit-cycle is stable for $b < 0$ (Fig. 292) and unstable for $b > 0$ (Fig. 293). From (44) we see that K vanishes with b and, when $\xi = 0$

($RC = MS/2$), the limit-cycle goes to infinity. The minimum value is $K = |b|$ (then for $b < 0$ $\sin \xi = -1$). Thus we see that a limit-cycle appears instantaneously. When it appears (when $RC = MS$),

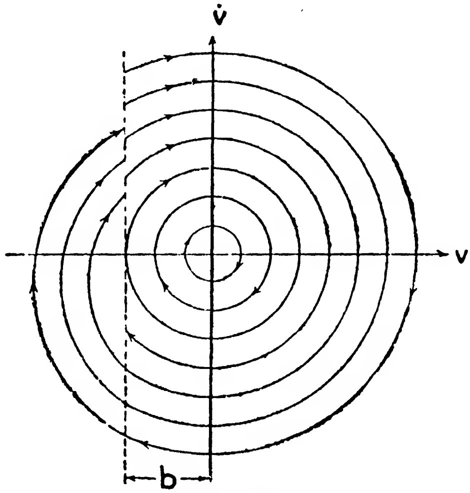


Fig. 294.

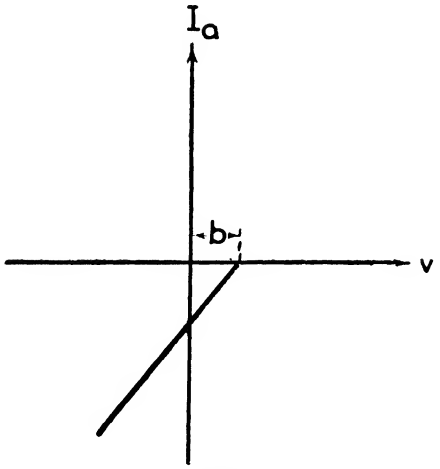


Fig. 295.

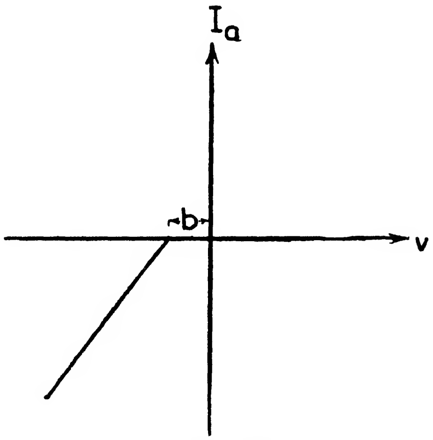


Fig. 296.

the amplitudes must be smaller than b, i.e. we have a center within the circle of radius b (Fig. 294).

The above results may be easily applied to the "converse characteristics" shown in Fig. 295 and 296. For simplicity we let $-b$ be the distance from the origin to the bend in the curves.

The equations have the same form as above, but their ranges of application are different:

$$
(50) \qquad \begin{cases} v > -b: & \ddot{v} + v = -R\sqrt{\dfrac{C}{L}}\,\dot{v}, \\ v < -b: & \ddot{v} + v = -R\sqrt{\dfrac{C}{L}}\,\dot{v} + \dfrac{MS}{\sqrt{LC}}\,\dot{v}. \end{cases}
$$

Substituting $v = -v_1$, we have

$$
v_1 < b: \qquad \ddot{v}_1 + v_1 = -R\sqrt{\frac{C}{L}}\,\dot{v}_1,
$$

$$
v_1 > b: \qquad \ddot{v}_1 + v_1 = -R\sqrt{\frac{C}{L}}\,\dot{v}_1 + \frac{MS}{\sqrt{LC}}\,\dot{v}_1,
$$

which is our previous system (42). Hence the preceding results apply directly to this case.

The preceding cases are interesting, in spite of the fact that characteristics limited on one side cannot exist; for when considering a bend on the characteristic the farther portions of the characteristic may not matter and in the neighborhood of the bend the situation is as considered here.

§6. INFLUENCE OF THE GRID CURRENT ON THE PERFORMANCE OF VACUUM TUBES

In previous discussions of vacuum tube circuits we have always neglected the grid current. This assumption, which considerably

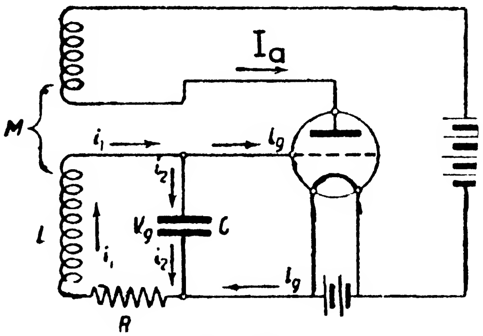


FIG. 297.

simplifies the problem, is often but not always satisfied. Without it the problem generally becomes more complicated in that the order of the differential equation describing the circuit is increased. Therefore, when we consider problems involving a single second-order differential equation, we are not in general taking account of the grid current. In certain cases it is possible, however, to take account of it

without increasing the order of the equations, e.g. in the case of a vacuum tube with a tuned grid circuit (Fig. 297).

We shall neglect the reaction and assume for simplicity that the characteristics of the plate and grid currents are third-degree polynomials in the grid voltage V_g:

$$\begin{aligned} I_a &= I_{a0} + S_1 V_g + S_2 V_g^2 - S_3 V_g^3, \\ i_g &= i_{g0} + p_1 V_g + p_2 V_g^2 + p_3 V_g^3. \end{aligned}$$

By Kirchhoff's law:

$$i_g = i_1 - i_2; \qquad V_g = \frac{1}{C}\int i_2\,dt; \qquad i_2 = C\frac{dV_g}{dt};$$

$$L\dot{i}_1 + Ri_1 + \frac{1}{C}\int i_2\,dt - M\dot{I}_a = 0$$

Eliminating i_1 from these equations, we find

$$L(\dot{i}_g + \dot{i}_2) + R(i_g + i_2) + V_g - M\dot{I}_a = 0$$

or

$$\begin{aligned} L\left(p_1\frac{dV_g}{dt} + p_2\frac{d(V_g^2)}{dt} + p_3\frac{d(V_g^3)}{dt} + C\frac{d^2V_g}{dt^2}\right) \\ + R\left(i_{g0} + p_1V_g + p_2V_g^2 + p_3V_g^3 + C\frac{dV_g}{dt}\right) \\ + V_g - M\left(S_1\frac{dV_g}{dt} + S_2\frac{d(V_g^2)}{dt} - S_3\frac{d(V_g^3)}{dt}\right) = 0 \end{aligned}$$

and, finally,

$$\begin{aligned} \frac{d^2V_g}{dt^2} + \left(\frac{R}{L} - \frac{MS_1}{LC} + \frac{p_1}{C}\right)\frac{dV_g}{dt} + \left(\frac{1}{LC} + \frac{Rp_1}{LC}\right)V_g \\ - \left(\frac{MS_2}{LC} - \frac{p_2}{C}\right)\frac{d(V_g^2)}{dt} + \left(\frac{MS_3}{LC} + \frac{p_3}{C}\right)\frac{d(V_g^3)}{dt} \\ + \frac{R}{LC}i_{g0} + \frac{Rp_2}{LC}V_g^2 + \frac{Rp_3}{LC}V_g^3 = 0. \end{aligned}$$

Let us set

$$\frac{R}{L} - \frac{MS_1}{LC} + \frac{p_1}{C} = -\alpha_1; \qquad \frac{MS_2}{LC} - \frac{p_2}{C} = \beta_1; \qquad \frac{MS_3}{LC} + \frac{p_3}{C} = \gamma_1;$$

$$\frac{1}{LC} = \omega_0^2; \qquad \omega_0^2(1 + Rp_1) = \omega_1^2; \qquad \frac{Rp_2}{LC} = n;$$

$$\frac{Rp_3}{LC} = m; \qquad \frac{Ri_{g0}}{LC} = p.$$

With adequate assumptions about the magnitude of the coefficients, our differential equation can be easily reduced to the form:

$$\ddot{x} + x = \mu f(x,\dot{x})$$

where x is dimensionless and μ is a small parameter. Since the van der Pol and Poincaré theories were developed on the basis of this parameter, we can obtain general formulas representing the periodic solutions, the correction for the frequency in first approximation, etc.

However, we shall not use these formulas now, and rather obtain the results with a minimum of calculations. We assume $V_g = K \sin \Omega t$ and introduce the "divergence" a^2,

$$a^2 = \Omega^2 - \omega_1^2,$$

and shall assume that α_1, β_1, γ_1, m, n, p, and a^2 are small (of the order of μ) with respect to the frequency ω_0. The equation of motion then takes the form:

$$\frac{d^2V_g}{dt^2} + \Omega^2 V_g = \alpha_1 \frac{dV_g}{dt} + \beta_1 \frac{d(V_g^2)}{dt} - \gamma_1 \frac{d(V_g^3)}{dt} - nV_g^2 - mV_g^3 - p + a^2 V_g. \tag{51}$$

To determine the amplitude and the correction for the frequency, we insert $V_g = K \sin \Omega t$ at the right and annul the resonance terms of the equation. Thus we obtain

$$K\alpha_1\Omega - \frac{3\gamma_1 K^3 \Omega}{4} = 0, \qquad -m\frac{3}{4}K^3 + a^2K = 0,$$

which we can solve for K^2 and a^2 as:

$$K^2 = \frac{\alpha_1}{\frac{3}{4}\gamma_1}, \qquad a^2 = \frac{3}{4}K^2 m = \frac{m\alpha_1}{\gamma_1}. \tag{52}$$

For the frequency we find

$$\Omega^2 = \frac{1}{LC} + \frac{Rp_1}{LC} + \frac{m\alpha_1}{\gamma_1}.$$

The process $V_g = K \sin \Omega t$ is stable if the constant term of the Fourier expansion of the derivative of the right side of (51) with respect to $\dot{V}_g$ is negative (see (38)), i.e. if the constant term of the expansion of $\alpha_1 + 2\beta_1 V_g - 3\gamma_1 V_g^2$ is negative. This means:

$$\alpha_1 - \frac{3}{2}\gamma_1 K^2 < 0, \qquad K^2 > \frac{1}{2} \cdot \frac{\alpha_1}{\frac{3}{4}\gamma_1}.$$

This condition is always satisfied because of (52), so the periodic motion is always stable. Finally, the condition for self-excitation of the circuit is $\alpha_1 > 0$, or

$$\frac{R}{L} + \frac{p_1}{C} - \frac{MS_1}{LC} < 0,$$

i.e. with regard to self-excitation the effect of the grid current is that of an additional load on the circuit.

§7. Branch Points for a Self-Oscillatory System Close to a Linear Conservative System

We shall now discuss systems with equations of motion of type

$$\ddot{x} + x = \mu f(x,\dot{x},\lambda) \tag{53}$$

where x is the coordinate of the system (displacement, voltage, etc.), μ is the small parameter characterizing the approximation to a linear conservative system, λ is a second parameter (for example an inductance) whose influence on the system we shall study, and $f(x,\dot{x},\lambda)$ is a non-linear function defined by the physical nature of the resistances and the arrangements of the power supply. By Poincaré's method we have shown that for small $\mu \neq 0$ only isolated closed paths, similar to circles, remain in the plane. The radii K of these curves are given by

$$\frac{B(2\pi)}{2\pi} = -\frac{1}{2\pi}\int_0^{2\pi} f(K\cos u, -K\sin u,\lambda)\sin u\,du = 0. \tag{54}$$

The other paths will be spirals, differing little from circles for small μ. The periodic motions corresponding to the isolated closed paths are stable if

$$\frac{1}{2\pi}\int_0^{2\pi} f_y(K\cos u, -K\sin u,\lambda)\,du < 0. \tag{55}$$

We shall now show that the character of the stationary motions depends on the parameter in the same way as in the situation of Chap. II, §5.

Set $K^2 = \rho$ and write the left side of (54), after multiplication by $2\sqrt{\rho}$, in the form

$$\Phi(\rho,\lambda) = -\frac{1}{\pi}\int_0^{2\pi} f(\sqrt{\rho}\cos u, -\sqrt{\rho}\sin u,\lambda)\sqrt{\rho}\sin u\,du.$$

As a consequence

$$\bar{\Phi}_\rho(\rho,\lambda) = -\frac{1}{2\pi\sqrt{\rho}}\int_0^{2\pi}(f_x\sqrt{\rho}\sin u\cos u + f_y\sqrt{\rho}\cos^2 u)\,du$$
$$+\frac{1}{2\pi}\int_0^{2\pi}f_y\,du - \frac{1}{2\pi\sqrt{\rho}}\int_0^{2\pi}f\sin u\,du = \frac{1}{2\pi\sqrt{\rho}}(f\cos u)_0^{2\pi}$$
$$+\frac{1}{2\pi\sqrt{\rho}}\int_0^{2\pi}f\sin u\,du + \frac{1}{2\pi}\int_0^{2\pi}f_y\,du - \frac{1}{2\pi\sqrt{\rho}}\int_0^{2\pi}f\sin u\,du,$$

and so finally

$$\bar{\Phi}_\rho(\rho,\lambda) = \frac{1}{2\pi}\int_0^{2\pi}f_y(K\cos u, -K\sin u, \lambda)\,du.$$

Consequently, (54) and (55) can be written:

$$\bar{\Phi}(\rho,\lambda) = 0, \qquad \bar{\Phi}_\rho(\rho,\lambda) < 0.$$

These conditions are completely analogous to those (Chap. II, §5) for the equilibrium state of a conservative system; the only difference is that we must investigate, instead of the coordinates of the singular points $\bar{x}_1, \bar{x}_2, \cdots, \bar{x}_s$, the squares of the amplitudes of the stationary motions $\bar{\rho}_1, \bar{\rho}_2, \cdots, \bar{\rho}_s$, which include the limit-cycles (in this case similar to circles) and the singular point $\rho = 0$.

Thus the stationary motions depend on the parameter in the same way that the equilibrium states did before. We obtain sets of stationary motions (instead of equilibrium states) which preserve their stability or instability up to the branch points. We have seen that the branch points have an important physical meaning; they correspond to the values of the parameter at which qualitative changes occur, e.g. occurrence or disappearance of oscillations, etc. The stationary motions described here, like the equilibrium states of the conservative systems, form a closed system of elements, between which there occurs an "exchange of stability."

Before investigating a concrete example we write down our formulas for determining the amplitudes and stability in the case when $f(x,\dot{x},\lambda)$ is of the form $f(x,\lambda)\dot{x}$, with $f(x,\lambda)$ a power series in x:

$$f(x,\lambda) = a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots,$$

whose coefficients $a_0, a_1, \cdots$ are functions of λ. By simple integration, we find

$$(56)\qquad \begin{cases} \bar{\Phi}(\rho,\lambda) = \left(a_0\rho + \dfrac{a_2\rho^2}{4} + \dfrac{a_4\rho^3}{8} + \cdots\right) \\ \bar{\Phi}'_\rho(\rho,\lambda) = \left(a_0 + \dfrac{a_2\rho}{2} + \dfrac{3}{8}a_4\rho^2 + \cdots\right). \end{cases}$$

§8. Application of the Theory of Branch Points to the Performance of Vacuum Tubes

We shall discuss soft and hard excitation of vacuum tubes. To avoid repetition we consider a vacuum tube oscillator with a tuned plate circuit (Fig. 298). As usual we simplify the situation by neglecting the grid current and plate reaction. The equation of the current in the oscillating circuit is

$$LC\ddot{i} + RC\dot{i} + i = i_a \tag{57}$$

where $i_a = \phi(V)$ is the plate current depending on the grid voltage V,

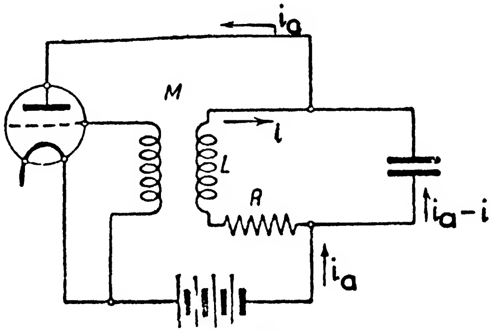


Fig. 298.

where $V = M\dot{i}$. We shall assume that the characteristic of the tube is expressed sufficiently accurately by a polynomial of degree five:

$$i_a = \phi(V) = \alpha_0 V + \beta_0 V^2 + \gamma_0 V^3 + \delta_0 V^4 + \epsilon_0 V^5.$$

We introduce new variables:

$$\tau = t/\sqrt{LC}, \qquad x = V/V_0$$

where V_0 is a constant voltage, e.g. saturation voltage, and choose $\mu = \beta_0 V_0 \sqrt{L/C}$ (which is also without dimension). Then (57) can be reduced to

$$\frac{d^2x}{d\tau^2} + x = \mu \frac{M}{L} \{\alpha + 2x + \gamma x^2 + \delta x^3 + \epsilon x^4\} \frac{dx}{d\tau},$$

where

$$\alpha = \frac{M\alpha_0 - CR}{\beta_0 V_0 M}, \qquad \gamma = \frac{3\gamma_0 V_0}{\beta_0}, \qquad \delta = \frac{4\delta_0 V_0^2}{\beta_0}, \qquad \epsilon = \frac{5\epsilon_0 V_0^3}{\beta_0}$$

are also dimensionless parameters and not small.

By (56) the conditions for the amplitude and stability (up to a positive factor) can be written:

$$\bar{\Phi}(\rho, M) = (M\alpha_0 - RC)\rho + \tfrac{3}{4}\gamma_0 V_0^2 M \rho^2 + \tfrac{5}{8}\epsilon_0 V_0^4 M \rho^3 = 0 \tag{58}$$

$$\bar{\Phi}_\rho(\rho, M) = (M\alpha_0 - RC) + \tfrac{3}{2}\gamma_0 V_0^2 M \rho + \tfrac{15}{8}\epsilon_0 V_0^4 M \rho^2 < 0. \tag{59}$$

The mutual inductance M is the parameter whose influence we wish to investigate. We must then construct the branch point diagrams ρ vs. M for the soft and hard oscillations. We shall consider only the region $M > 0$ corresponding to the normal direction of the turns of the coil of the feedback coupling. It should be remembered that only the values $\rho \geqslant 0$ have physical meaning; they correspond to the real amplitudes of stationary solutions.

To simplify the discussion we shall always make such assumptions about the coefficients of i_a as to have the simplest mathematical idealization.

1. Soft generation of oscillations. The assumptions $\alpha_0 > 0$, $\gamma_0 < 0$, $\epsilon_0 = 0$ are simplest in this case. Letting

$$\tfrac{3}{4}\gamma_0 V_0^2 = -\alpha \quad (\alpha > 0),$$

we can write

$$\bar{\Phi}(\rho,M) = \{M\alpha_0 - RC - \alpha M\rho\}\rho \tag{60}$$

This shows that in the ρ,M-plane the branch point diagram breaks up into a straight line $\rho = 0$ and a hyperbola: $M\alpha_0 - RC - \alpha M\rho = 0$.

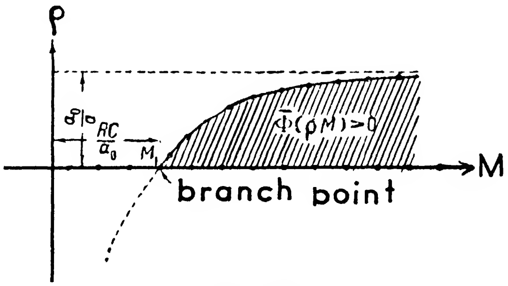


FIG. 299.

Condition (59) allows us to isolate the regions in the ρ,M-plane where $\bar{\Phi}(\rho,M) > 0$ and, by the general rules of Chap. V, §8, to determine the stable (black circles) and unstable (light circles) portions of the scheme (Fig. 299).

The branch point of M is $M_1 = RC/\alpha_0$ in which the arcs corresponding to the straight line and hyperbola intersect. The straight line is stable up to $M = M_1$, at which point the hyperbola becomes stable. Now let us consider the phase plane for different values of M. If $M < M_1$ there is only one stable stationary state, a stable focus (Fig. 300). The representative point moving along any of the spirals will finally reach the vicinity of the stable singular point. When M passes through M_1, a stable limit-cycle separates from the singular point (Fig. 301). The representative point previously at the equi-

librium state is transformed into a limit-cycle, because the equilibrium is unstable for $M > M_1$. In the language of physics, the system experiences oscillations and self-excitation occurs. When M increases, the radius of the limit-cycle increases and tends to the asymptotic

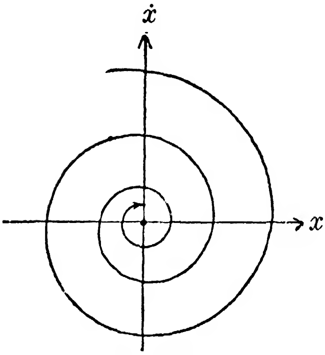


FIG. 300.

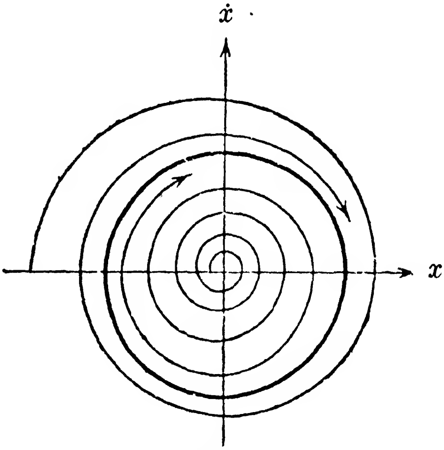


FIG. 301.

value corresponding to $\rho = \alpha_0/\alpha$. When M decreases, the system behaves in a converse manner, the limit-cycle shrinks, the representative point "follows" the limit-cycle; and when $M = M_1$ (when the limit-cycle becomes a point), the representative point returns to the origin, which becomes then a stable focus.

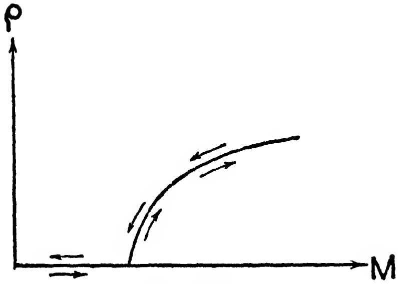


FIG. 302.

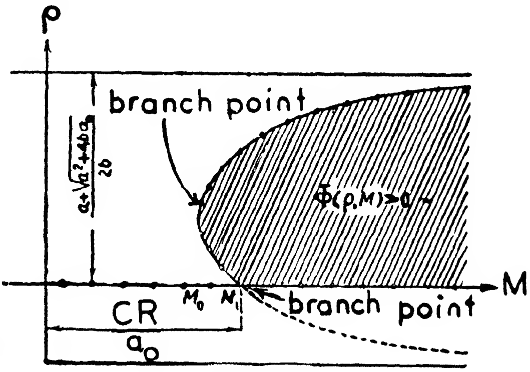


FIG. 303.

The apparatus measuring the amplitude of the oscillations as a function of M will register a smooth ("soft") curve. The passage from the state of rest to stationary oscillations and conversely will be continuous (without a jump), with a gradual change of the amplitude (Fig. 302).

2. Hard generation of oscillations. The assumptions $\alpha_0 > 0$, $\gamma_0 > 0$, $\epsilon_0 < 0$ will exhibit the main characteristics of hard generation. Setting

$$\tfrac{3}{4}\gamma_0 V_0^2 = a \quad (a > 0); \qquad \tfrac{5}{8}\epsilon_0 V_0^4 = -b \quad (b > 0),$$

we can write

$$\bar{\Phi}(\rho,M) = (M\alpha_0 - RC + aM\rho - bM\rho^2)\rho. \tag{61}$$

In the (ρ,M)-plane the diagram consists of the line $\rho = 0$ and the curve of third order,

$$M\alpha_0 - RC + aM\rho - bM\rho^2 = 0. \tag{62}$$

The approximate situation of these curves, the regions where $\bar{\Phi}(\rho,M)$

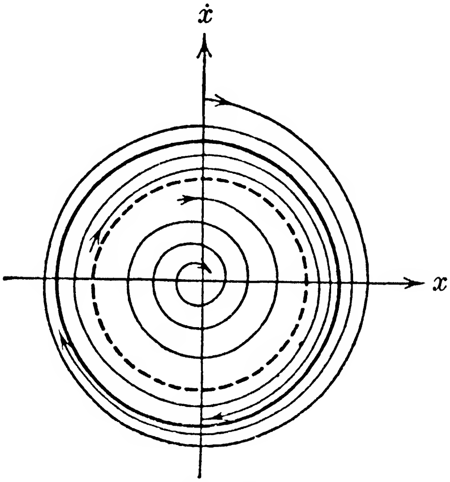


FIG. 304.

> 0, and the stable (black circles) and unstable (light circles) portions of the curves are shown in Fig. 303.

There are two positive branch points

$$M_0 = \frac{RC}{\alpha_0 + \dfrac{a^2}{4b}}, \qquad M_1 = \frac{RC}{\alpha_0}.$$

M_0 corresponds to the coming together and M_1 to the intersection of two arcs.

Now consider the picture in the phase plane. When $0 < M < M_1$ we have, as in the previous case, only one stationary motion, a stable focus at the origin (Fig. 300). When M passes through M_0, we obtain a pair of limit-cycles (Fig. 304). The larger is stable, the smaller

unstable, and the singular point at the origin remains stable. When M increases further, the stable cycle increases and the unstable one decreases (Fig. 305) until, when $M = M_1$, it coincides with the singular point, which then becomes unstable (Fig. 306). The radius of the stable cycle tends asymptotically to the positive root of the equation

$$\alpha_0 + a\rho - b\rho^2 = 0$$

obtained from (62) by dividing by M and making $M \to \infty$. This positive root is

$$\rho = \frac{a + \sqrt{a^2 + 4b\alpha_0}}{2b}$$

We can ask what the apparatus measuring the amplitude of the oscil-

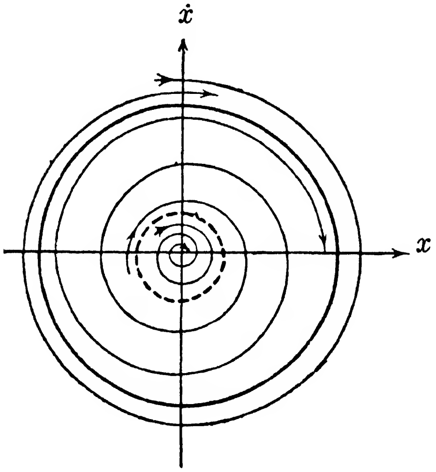


FIG. 305.

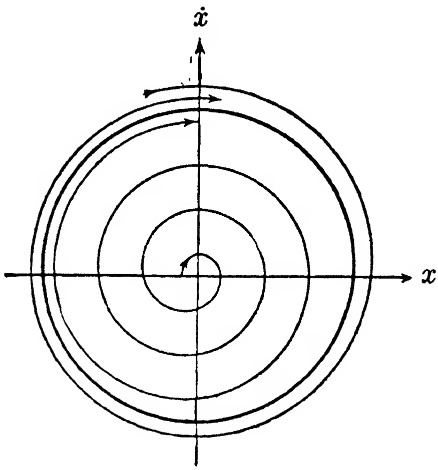


FIG. 306.

lations will register. To answer this we must examine the behavior, for small M, of a point in the neighborhood of the equilibrium state. Obviously this point will remain there as long as the state remains stable, i.e. until $M = M_1$. That two limit-cycles appear at $M = M_0$ does not matter for our point because it does not affect the stability of the equilibrium.

For $M > M_1$, the singular point is unstable; our point "jumps" when M passes through $M = M_1$, then moves according to the "instructions given" by the paths and consequently comes to a stable limit-cycle on which it stays for all larger M. When M decreases, we obtain a completely different picture. Our point remains on the limit-cycle for $M > M_1$. When $M = M_1$ the equilibrium remains

stable because this does not change the character of the limit-cycle along which the point is moving. When M passes through M_0, the point following the path passes to an equilibrium state and remains there for all smaller M.

The apparatus measuring the amplitude of the grid voltage will register jumps at M_1 when M increases and at M_0 when M decreases.

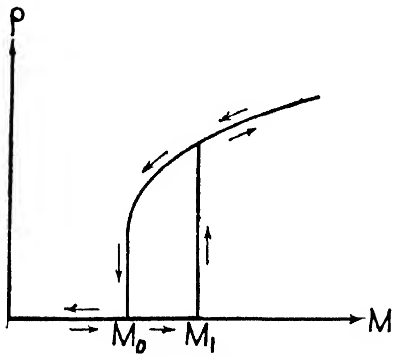


FIG. 307.

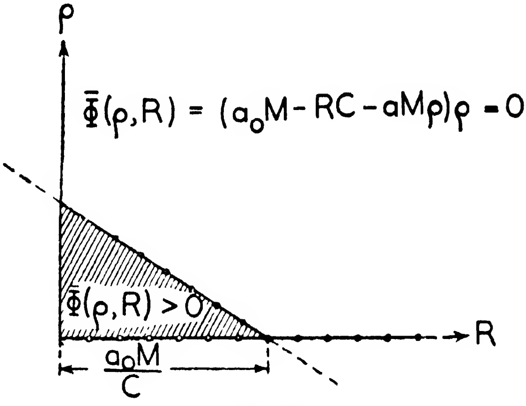


FIG. 308.

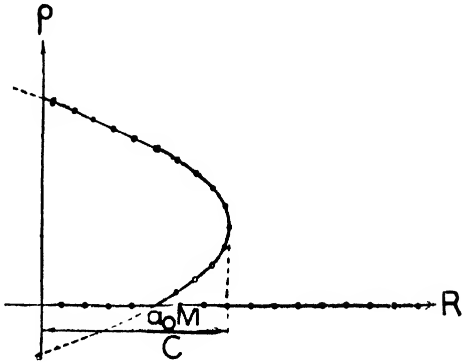


FIG. 309.

The difference between the situation for increasing and decreasing M means that our process has an irreversible "hysteresis-like" character (Fig. 307).

We have obtained branch point diagrams corresponding to soft and to hard oscillations in terms of a particular parameter, the coefficient of self-induction. Analogous diagrams can be obtained for other parameters characterizing our system. Figs. 308 and 309 represent branch point diagrams ρ vs. R, where R is the ohmic resistance, for cases of soft and hard working conditions. The corresponding relations can be derived easily from (60) and (61).

APPENDIX A

Structural Stability

There has been introduced in Chapter V a notion of structural stability for the case when the right-hand sides of the differential equations are polynomials. We return to the question here from a somewhat broader standpoint and with the main object of giving additional mathematical information (without proofs). Complementary references are: Andronow-Pontrjagin, Leontowitch-Maier, both in *Comptes Rendus* (Doklady), Acad. Sc. U.S.S.R., Vol. 14, 1937.

Let us suppose then that we are dealing with an autonomous system

$$(1) \qquad \dot{x} = P(x,y), \qquad \dot{y} = Q(x,y)$$

where this time P and Q are merely analytic within and on the boundary of a region Ω. The boundary is supposed to be a simple closed curve.

The vector (P,Q) is referred to as the *velocity vector:* understood of the representative point $M(x,y)$ of the system. An arc λ is said to be *without contact* whenever the velocity vector is neither zero nor tangent to λ at any point of λ. Similarly if these properties hold for every point of a single closed curve γ we say that γ is a *cycle without contact.* The terms and concepts go back to Poincaré who utilized them extensively in his fundamental paper.

It is required to isolate a class of systems which, above all, have a rather permanent phase portrait within the region Ω. The physical necessity for this is fairly clear; in physical problems one never knows exactly what the functions P and Q are and so one will naturally exclude systems which are affected by ever so slight a modification of these functions. Moreover in all physical problems the basic magnitudes (current, voltage, displacement, velocity) remain finite. One is only concerned with finite regions and regions which do not vary throughout the discussion. These observations justify the following conditions to be imposed upon the system:

I. The boundary G of the region of analyticity Ω of P and Q is a cycle without contact.

One must select Ω large enough to include all the "practically" reachable values of x and y (all the practically reachable points of the phase plane).

II. Corresponding to Ω there exists an $\epsilon > 0$ such that if at all points of Ω and G the functions p,q are analytical and

$$|p(x,y)| < \epsilon, \qquad |q(x,y)| < \epsilon \tag{2}$$

then the modified system

$$\dot{x} = P + p, \qquad \dot{y} = Q + q \tag{3}$$

has the same phase portrait, qualitatively, as the initial system (1).

A system (1) which possesses both properties I and II is said to be *structurally stable in* Ω. This imposes a number of far-reaching restrictions on the system, and hence restricts very greatly the phase portrait in Ω. A few of the restrictions are:

A. There are only simple singular points in Ω, that is to say only points in which the real parts of the characteristic roots are $\neq 0$.

B. If γ is a limit-cycle in Ω with related periodic solution $x = f(t)$, $y = g(t)$ and period T then the characteristic exponent

$$h(\gamma) = \frac{1}{T}\int_0^T [P_x + Q_y]\,dt, \qquad x = f(t), \qquad y = g(t)$$

does not vanish.

C. There exists no separatrix in Ω joining two saddle points. That is to say if δ is a separatrix issued from a saddle point when followed in one direction then it cannot tend to another saddle point when followed in the other direction.

D. The following converse property holds: Let (1) be analytic in a region Ω and on its boundary G and let G be a cycle without contact for the system. Moreover let properties A,B,C hold relative to Ω. Then the system (1) is structurally stable relative to Ω.

There are also noteworthy consequences of structural stability for the limits towards which paths tend in the region Ω. Let us call an *ordinary* path one which is neither a singular point nor a limit-cycle. We observe at once that when the system is structurally stable then:

E. The only possible singular points in Ω are nodes, foci, and saddle points—the types considered in Chap. I.

F. The only possible limit-cycles in Ω are those which are stable on both sides or unstable on both sides.

G. The ordinary paths in Ω fall under the following categories:

$$\text{Separatrices}\begin{cases}\text{leaving an unstable focus or node,}\\ \text{entering the region } \Omega,\\ \text{tending to a stable focus or node,}\\ \text{tending to a stable limit cycle.}\end{cases}$$

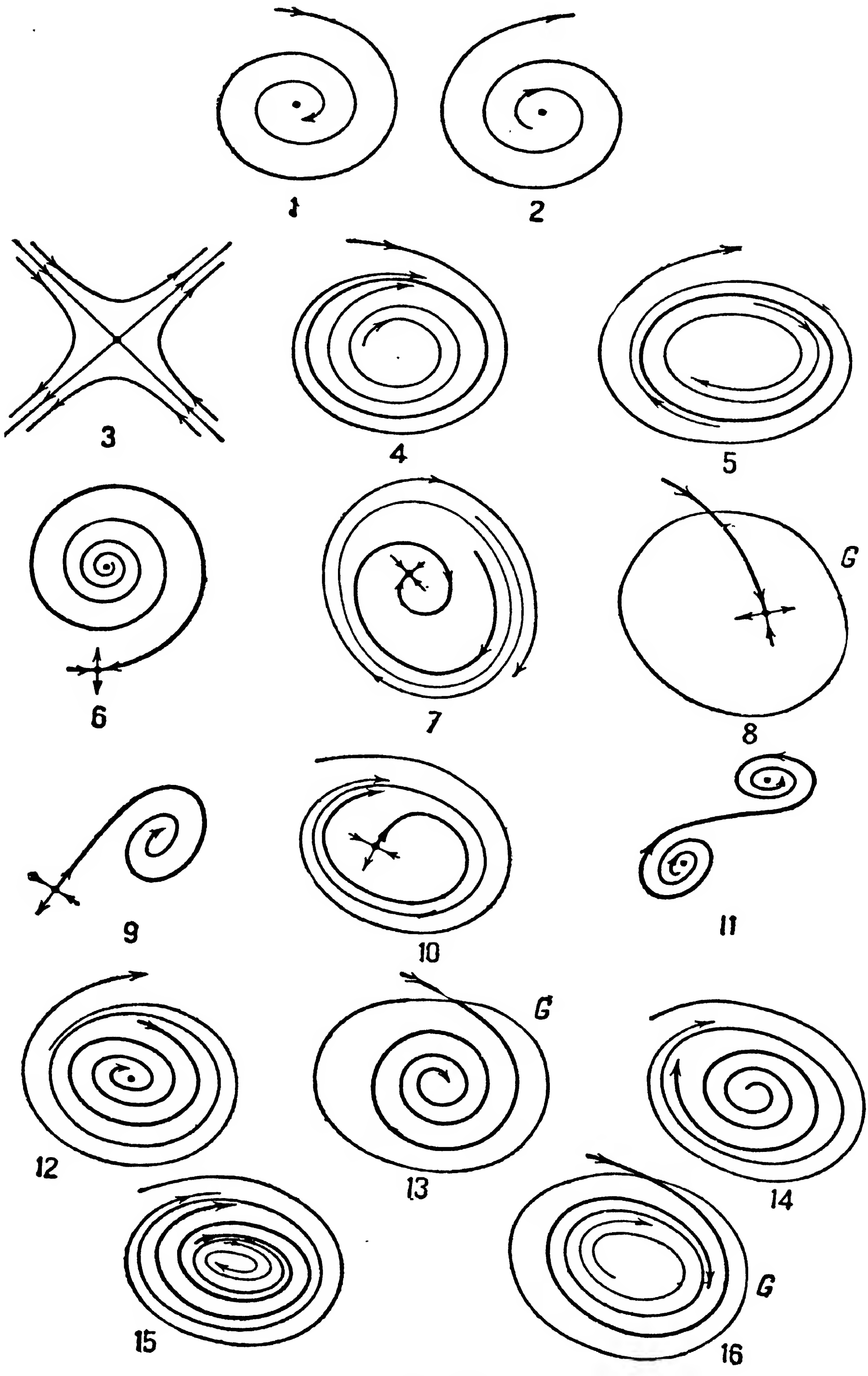


FIG. 310.

Paths tending to a stable focus, node or limit-cycle $\begin{cases}\text{leaving an unstable focus or node,}\\ \text{leaving an unstable limit cycle,}\\ \text{entering } \Omega.\end{cases}$

The various possibilities are illustrated in Fig. 310.

H. Any ordinary path passing sufficiently close to an ordinary path γ in Ω behaves exactly like γ in the sense described in G.

APPENDIX B

Justification of the van der Pol Approximations

The van der Pol method consists essentially in the following: given a system with terms periodical in the time, these terms are dropped making the system autonomous. Thus the system (2) of Chap. IX is replaced by the system (3) in the same chapter and it is to be shown that for a sufficiently large time T, the solutions of the two systems with the same initial conditions remain quite close together provided that the parameter is sufficiently small. For simplicity we shall merely consider a single equation. The systems of several equations may be treated in similar manner. (The original Russian contains a rather long proof attributed to Mandelstam and Papalexi and gives a reference to Fatou. The short proof here given is due to the editor.)

We shall prove a somewhat more general result which may be stated thus: Consider the equations

$$\dot{a} = \mu f(a) \tag{1}$$

$$\dot{a} = \mu(f(a) + g(a,t)) \tag{2}$$

where f and g are analytic for $t > \tau$ and $|a| < A$ and suppose that for I_α: $0 \leqslant \mu \leqslant \alpha$, (1) has an analytic solution $a(t,\mu)$ such that $a(t_0,\mu) = a_0$, $t_0 > \tau$, where for all values considered $|a(t,\mu)| < A$. Then given any ϵ, T positive one can choose an η so small that (2) has a solution $\bar{a}(t,\mu)$ such that $\bar{a}(t_0,\mu) = a_0$ and that for $0 \leqslant \mu < \eta$, $t_0 \leqslant t \leqslant t_0 + T$:

$$|\bar{a}(t,\mu) - a(t,\mu)| < \epsilon. \tag{3}$$

To prove this property consider the auxiliary equation

$$\dot{b} = \mu f(b) + \nu g(b,t),$$

and take any μ_0 in I_α. By a well known theorem due to Poincaré (see Lefschetz, *Lectures on differential equations,* p. 35) there is a solution $b(t,\mu,\nu)$ such that $b(t_0,\mu,\nu) = a_0$ and numbers $\eta,\zeta > 0$ such that

$$|b(t,\mu,\nu) - a(t,\mu_0)| < \tfrac{1}{2}\epsilon$$

for $t_0 \leqslant t \leqslant t_0 + T$, and

$$I'_\eta:\ \mu_0 - \eta < \mu < \mu_0 + \eta, \qquad I''_\zeta:\ 0 \leqslant \nu < \zeta.$$

Again applying Poincaré's theorem, this time to $a(t,\mu)$, one may choose η so small that for μ in I'_η and $t_0 \leqslant t \leqslant t_0 + T$ we have

$$|a(t,\mu) - a(t,\mu_0)| < \tfrac{1}{2}\epsilon.$$

Hence for the same range of μ,ν,t we will have

$$|b(t,\mu,\nu) - a(t,\mu)| < \epsilon. \tag{4}$$

The number η depends actually upon μ_0. However since μ_0 varies over a closed interval, namely I_α, by the Heine-Borel theorem, one may cover I_α with a finite number of intervals $I'_{\eta_1}, \cdots, I'_{\eta_s}$ with associated $I''_{\zeta_1}, \cdots, I''_{\zeta_s}$ such that (4) holds for μ in I'_{η_i} and ν in I''_{ζ_i}. Let ρ be the least of the numbers η_i,ζ_i. Then (4) will hold for $0 \leqslant \mu < \rho$, $0 \leqslant \nu < \rho$. Hence for $0 \leqslant \mu < \rho$, $\bar{a}(t,\mu) = b(t,\mu,\mu)$ will be a solution of (2) such that (3) holds for $t_0 \leqslant t \leqslant t_0 + T$. This is the result which we had in view.

The extension to any number n of equations is automatic; one has only to consider $a,\bar{a},b,f(a),g(a,t)$ as n-dimensional vectors, and interpret inequalities such as $|b - a| < \epsilon$ as referring to vector norms. (For details regarding these concepts see Lefschetz, *op. cit.*, p. 11.)

APPENDIX C

The van der Pol Equation for Arbitrary Values of the Parameter[1]

Let us take the van der Pol equation in the form

$$\ddot{x} + \mu(x^2 - 1)\dot{x} + x = 0 \tag{1}$$

where x represents a current. The considerations of Chap. IX were confined to small values of the parameter. We have also mentioned that Liénard has shown that whatever μ there is a unique periodic solution $\bar{x}(t)$. There is a long standing surmise regarding the nature of $\bar{x}(t)$ as $\mu \to +\infty$. The correctness of this surmise was established recently by Flanders and Stoker (*Studies in non-linear mechanics*, Institute of Mathematics and Mechanics, New York University, 1946). Their results in turn have been extended and given more precision by J. LaSalle who has obtained information valid for all values of μ. We merely propose here to describe his results as they apply to van der Pol's equation.

Following Flanders and Stoker it will be found more practicable to replace van der Pol's equation by one due to Rayleigh. This may be done as follows. If z denotes the charge we will have $x = \dot{z}$ and so

$$\dddot{z} + \mu(\dot{z}^2 - 1)\ddot{z} + \dot{z} = 0. \tag{2}$$

Integrating we have

$$\ddot{z} + \mu\left(\frac{\dot{z}^3}{3} - \dot{z}\right) + z = h.$$

As far as the periodic solutions are concerned we may replace z by $z + h$ and thus obtain Rayleigh's equation

$$\ddot{z} + \mu\left(\frac{\dot{z}^3}{3} - \dot{z}\right) + z = 0. \tag{3}$$

We replace (3) in the usual manner by

$$\dot{z} = x, \qquad \dot{x} = -\mu\left(\frac{x^3}{3} - x\right) - z \tag{4}$$

[1] The material contained in this Appendix is essentially based upon the work of J. LaSalle.

and follow this by the change of variables $z = \mu y$, $t = \tau/\mu$, $\mu = 1/\epsilon$, so that (4) is replaced by

$$(5) \qquad \frac{dx}{d\tau} = -\left(\frac{x^3}{3} - x\right) - y, \qquad \frac{dy}{d\tau} = \epsilon^2 x.$$

The periodic solution of (1) corresponds to the (unique) limit-cycle Γ of the system (5) and it is its position that we propose to discuss, with particular emphasis on its behavior as $\mu \to +\infty$, or which is the same as $\epsilon \to 0$.

The advantage of the various transformations performed is that the first relation in (5) is independent of ϵ, and thus provides us with a fixed "element" as ϵ varies. We shall utilize it to obtain some idea as to what happens to the limit cycle when $\epsilon \to 0$.

It is very advantageous to utilize the graph

$$\Delta: y = -\left(\frac{x^3}{3} - x\right)$$

which is shown in Fig. 311. It is in fact the isocline corresponding to infinite slope, i.e. the locus of the points where the paths of (5) have vertical tangents. Along Δ the velocity of the representative point is thus vertical, upwards to the right of the y-axis, downwards to its left. Away from the curve and for ϵ very small the velocity of the representative point is almost horizontal, towards the right below Δ and towards the left above it. It follows that wherever the representative point may start it will tend towards M on MD or towards M' on $M'D'$. Having reached however say M along Δ (or very near Δ) it cannot proceed along Δ towards the origin and must follow (more or less) the horizontal MA', then beyond A' the arc $A'M'$, etc. It will thus describe approximately the closed curve Γ_0 which is the surmised limiting curve of the limit-cycle Γ as $\epsilon \to 0$.

Flanders and Stoker established this limit property by constructing an *enclosure* (a band) containing Γ_0 whose width $\to 0$ with ϵ and showing that for ϵ sufficiently small Γ will have to be in the enclosure. LaSalle constructs for every ϵ an enclosure Ω containing Γ. It is also a band but instead of containing Γ_0 its inner boundary abuts on Γ_0 and here again the width of $\Omega \to 0$ with ϵ, and $\Omega \to \Gamma_0$, proving again that $\Gamma \to \Gamma_0$. However there is an enclosure for every ϵ and this leads to estimates for amplitude and period in terms of ϵ or μ whatever μ.

The construction of LaSalle is clear from Fig. 312. The arc ENL is a quadrant of an ellipse whose center is C and axes CL and CE.

The arc JHF is constructed by augmenting the ordinates along the arc $JABMG$ by $\epsilon\sqrt{a^2 - x^2}$. The band Ω is symmetrical with respect to the origin and limited to the left by the arc $ENLBMG$ and to the right by the arc $PJHF$. The inner boundary is only suitable when L

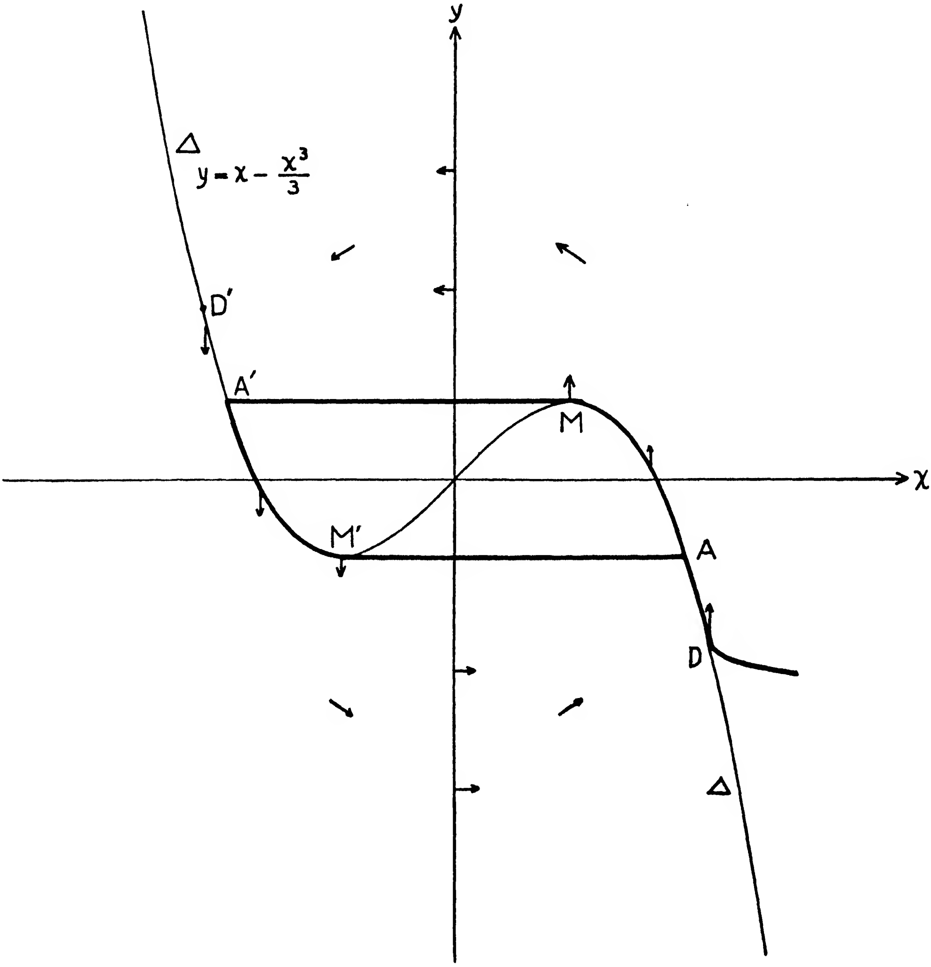


Fig. 311.

is not above B, i.e. when ϵ = slope $EL \leqslant$ slope $EB = \frac{2}{3} \cdot 1/\sqrt{3}$, i.e. for $\mu \geqslant 3\sqrt{3}/2 = 2.6 \cdots$. For all values of ϵ (or μ) one may choose for inner boundary the ellipse of center O whose horizontal axis is $OB = \sqrt{3}$ and whose vertical axis is $\epsilon\sqrt{3}$.

Both Flanders-Stoker and LaSalle show that there is a limit-cycle within their enclosures by proving that along the inner and outer boundaries the velocity vector of the representative point (the vector

whose components are the right-hand sides of (5)) points towards the interior of Ω. Since Ω contains no singular point, the Poincaré theorem (theorem IV, p. 209) asserts that it contains a limit-cycle and since there is only one, the unique limit-cycle Γ is contained in Ω.

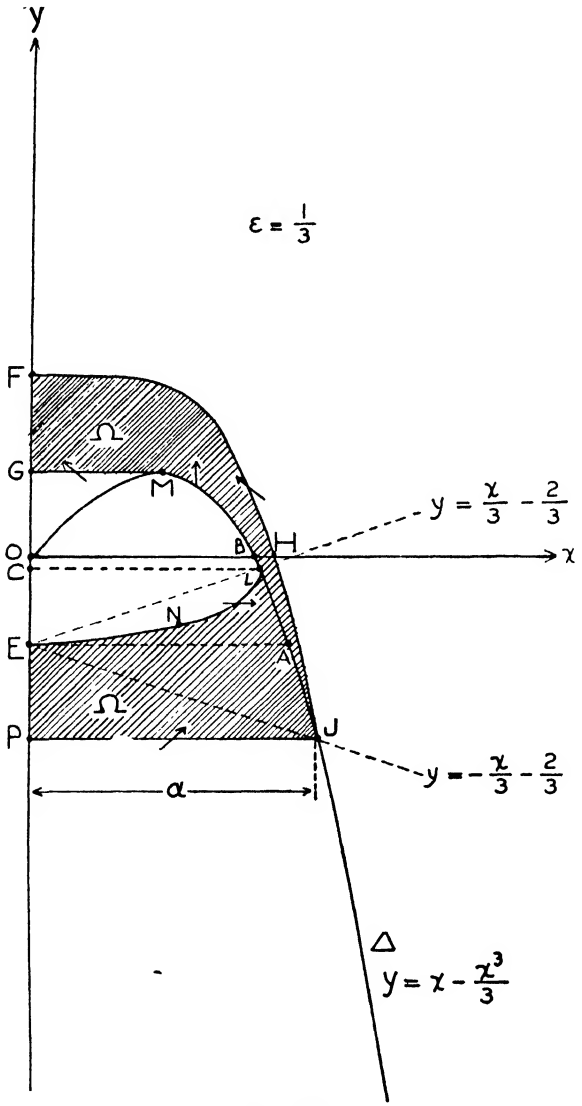


FIG. 312.

It is evident from the figure that the amplitude of x is less than PJ which is the largest x of an intersection point of $y + \frac{2}{3} = -\epsilon x$ with the curve Δ. This is also the largest positive root of

$$x^3 - 3(1 + \epsilon)x - 2 = 0.$$

For $\epsilon = 0$ this largest root is 2 and this is the amplitude of the limiting periodic motion as $\epsilon \to 0$. It was already given earlier by van der Pol.

By a refinement of the above enclosure method one may obtain for θ, the τ time that the representative point takes to describe Γ, van der Pol's relation

$$\lim_{\epsilon \to 0} \epsilon^2 \theta = 3 - 2 \log 2 = 1.61 \cdots$$

Since $t = \tau/\mu = \epsilon\tau$, the t time T that it takes to describe Γ, which is the period of the periodic motion will be given by

$$\lim \epsilon \mathrm{T} = 3 - 2 \log 2.$$

That is to say for ϵ small, or μ large

$$\mathrm{T} \doteq \frac{3 - 2 \log 2}{\epsilon} = 1.61\mu.$$

Roughly speaking then T is of the order of magnitude of μ.

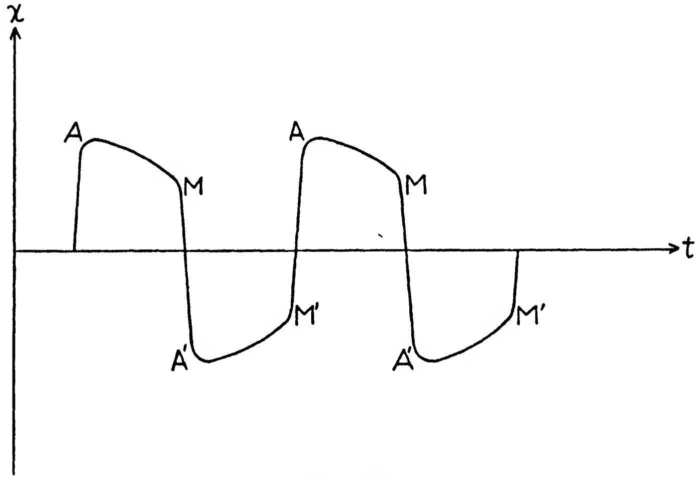


Fig. 313.

It is interesting to outline the graph of the periodic motion $\bar{x}(t)$. We infer from (4) that $\dot{x}$ is very small near Δ and very large away from it. Thus the parts of Γ which are almost horizontal will be described very rapidly and those very near Γ_0 (close to its curvilinear parts) will be described comparatively slowly. Thus the graph has the general aspect of Fig. 313 with an alternation of nearly horizontal and nearly vertical arcs. The resulting wave form is typical for relaxation oscillations.

For further details and a complete discussion the reader is referred to a paper of J. LaSalle (in a forthcoming number of the *Quarterly Journal of Applied Mathematics*).

Supplementary Reading List

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